CALCULUS OF VARIATIONS

CHAPTER I : INTRODUCTION AND INDIRECT METHODS (1D CASE)

I.1 INTRODUCTION

the calculus of variations is the study of minimizers or critical points of "functionals", which are functions defined in spaces of infinite dimensions, typically functional spaces.

This needs to adapt the notions of differential calculus. The stationarity of a functional $\mathcal{E}(u)$ is "simply" characterized by the equation

$$\boxed{\mathcal{E}'(u) = 0} \qquad (1)$$

which, in general, will be a ~~parti~~ PDE in u.

A GOAL of the calculus of variations is "to solve" such PDEs: more precisely to show that they actually have one or several solutions (or none!), study their properties, and possibly design numerical methods to compute these solutions or approximations.
A first important observation is that NOT ALL PDEs can be solved by a variational analysis: only PDEs which are "variationals", meaning that ~~they are eq~~ they are of the form (1) for a well chosen $\mathcal{E}$.

Why is it interesting?

- It provides sometimes a very simple tool for showing existence of (weak) solutions to a problem;
- Many PDEs come from problems in physics, mechanics etc and precisely from "variational" ~~prob~~ principles and are therefore critical points of some physical energy.

- many problems in the industry (finance) are designed as finding the 'best' state according to some criterion and their solution is precisely a minimizer, or maximizer, of this criterion.

- SOME EXAMPLES:

① Laplace equation $-\Delta u = 0$ characterizes the critical points of the "Dirichlet energy"

$$\mathcal{E}(u) = \int |\nabla u|^2 dx.$$

In this case, since the energy is convex, critical points and minimizers are the same.

② Horizontal elastic membrane subject to a vertical force f:

$$\min \left\{ \mathcal{E}(u) \mid u = 0 \text{ on } \partial\Omega \right\},$$

where

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx.$$

In this case equation (1) reads

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

~~③ [illegible]~~

③ Minimal surfaces, geodesics:

- $\mathcal{E}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx$
- $\mathcal{E}(u) = \int_0^1 |u'(x)|^2 dx$ with $u(0) = x_0$, $u(1) = x_1$ and $u(x) \in \Sigma$

For each of these problems, the natural questions are:

(i) is there existence of a solution? (ii) Uniqueness? (iii) How can it characterized?

1.2 THE ONE DIMENSIONAL CAS AND THE INDIRECT METHODS

We will start by anylizing the one dimensional case and the classical indirect method to solving variationals problems.
The first step of this method consists in deriving necessary conditions for a function to be a minimizer.
Among these, the principal condition is the vanishing of the first variation of the functional (to be minimized) at the candidate minimizer.
As a consequence of this, any smooth candidate has to satisfy the Euler equation.
The second step (and more difficult!) of the indirect method consists in obtaining a sufficient conditions for a critical point to be a minimizer.

1.2.1 EULER EQUATION AND THE NECESSARY CONDITIONS FOR OPTIMALITY

As mentioned above the principal necessary condition is the vanishing of the first variation ~~of~~ at the candidate minimizer.

Let us consider the functional of the form

$$\mathcal{E}(u) = \int_I \mathcal{L}(x, u(x), u'(x))\, dx \qquad (2)$$

which will be called variational ~~functional~~ integrals, $I = (a,b)$ and $\mathcal{L}(x,z,p)$ is called the Lagrangian of the variational pb and is defined on $\bar{I} \times \mathbb{R}^d \times \mathbb{R}^d$. We will assume for now that $\mathcal{L}$ is at least of class C^1. Then (2) is well defined for all $u \in C^1(\bar{I}; \mathbb{R}^d)$. Notice that often one can consider $\mathcal{L}$ $\mathcal{E}$ only in some "neighbourhood" of such function u. Then it is enough to assume $\mathcal{L} \in C^1(\mathcal{U})$ when $\mathcal{U}$ denotes some open set in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ containing $\{(x, u(x), u'(x)) \mid x \in \bar{I}\}$ (1-graph of u).

Now $\mathcal{E}(v)$ is defined for any $v \in C^1(\bar{I}; \mathbb{R}^d)$ ~~only~~ satisfying $\|v-u\|_{C^1} < \delta$ for some small enough $\delta > 0$.
It follows that the function

$$\Phi(\varepsilon) := \mathcal{E}(u + \varepsilon\varphi)$$

is defined for any choice of $\varphi \in C^1(\bar{I}; \mathbb{R}^d)$ and for $|\varepsilon| < \varepsilon_0$ where $\varepsilon_0 := \frac{\delta}{\|\varphi\|_{C^1}}$. Moreover $\Phi \in C^1((-\varepsilon_0, \varepsilon_0))$ and ~~we~~ deriving by ε we obtain:

$$\Phi'(0) = \int_I \left(\mathcal{L}_z(x,u,u') \cdot \varphi + \mathcal{L}_p(x,u,u') \cdot \varphi' \right) dx .$$

We set

$$\boxed{\delta\mathcal{E}(u,\varphi) := \Phi'(0)}$$

and we call $\delta\mathcal{E}(u,\varphi)$ the <u>first variation of $\mathcal{E}$ at u in the direction of φ</u>. We note that

$\delta\mathcal{E}(u,\varphi)$ is a <u>linear functional</u> of $\varphi \in C^1(\bar{I}; \mathbb{R}^d)$.

<u>definition</u> A function $u \in C^1(\bar{I}; \mathbb{R}^d)$ satisfying

$$(3) \qquad \int_I \left(\mathcal{L}_z(x,u,u') \cdot \varphi + \mathcal{L}_p(x,u,u') \cdot \varphi' \right) dx = 0$$

$\forall \varphi \in C_c^\infty(I; \mathbb{R}^d)$ is said to be a weak critical point of the functional $\mathcal{E}$.

<u>Rmk</u> We should call u weak-C^1 critical point as we will consider other kinds of weak extremals.

<u>Rmk</u> If $u \in C^1(\bar{I}; \mathbb{R}^d)$ then (3) is equivalent to

$$\delta\mathcal{E}(u,\varphi) = 0 \qquad \forall \varphi \in C_c^\infty(I; \mathbb{R}^d)$$

the critical points of $\mathcal{E}$ are critical points in the sense that the directional derivatives vanishes, that is

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathcal{E}(u+\varepsilon\varphi) - \mathcal{E}(u)] = 0$$

We easily have then

<u>Proposition</u> If $u \in \mathcal{C}^1(\bar{I}; \mathbb{R}^d)$ is a weak minimizer of $\mathcal{E}$, that is

$$\mathcal{E}(u) \leq \mathcal{E}(u+\varphi)$$

for all $\varphi \in \mathcal{C}^\infty_c(\bar{I}; \mathbb{R}^d)$ with $\|\varphi\|_{C^1} < \delta$, $0 < \delta \ll 1$, then u is a weak critical point of $\mathcal{E}$.

<u>Rmk</u> The converse is in general not true.

In order to show that the critical points satisfy our "optimality condition" we need some useful lemmas.

FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS

<u>Lemma</u> Suppose that $f \in \mathcal{C}(I)$ satisfies

$$\int_I f(x)\eta(x)\,dx = 0 \qquad \forall \eta \in \mathcal{C}^\infty_c(I). \qquad (4)$$

Then we have $f(x) = 0 \quad \forall x \in I$.

<u>proof</u> Let χ_0 be the characteristic function of some interval $I_0 = (x_0 - \delta, x_0 + \delta) \subset\subset I$, $\delta > 0$. Since $C^\infty_c(I)$ is dense in $L^2(I)$ with the respect to the L^2-norm we have from (4) that

$$\int_{I_0} f(x)\,dx = \int_I f(x)\,\chi_0(x)\,dx = 0$$

and therefore

$$\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = 0$$

letting $\delta \to 0$ we obtain $f(x_0) = 0 \quad \forall x_0 \in I$

$\square$

<u>Lemma</u> If $f \in L^1(I)$ satisfies (4), then $f(x) = 0$ a.e. on I.

<u>proof</u> Choose I_0 and x_0 as before and consider the piecewise linear function $\eta_\varepsilon \in C^0_c(I)$ defined by

$$\eta_\varepsilon(x) := \begin{cases} 1 & \text{on } I_0 \\ 0 & |x - x_0| \geq \delta + \varepsilon \\ \text{linear} & \text{in } (x_0 - \delta - \varepsilon, x_0 + \delta) \\ \text{linear} & \text{in } (x_0 + \delta, x_0 + \delta + \varepsilon) \end{cases}$$

$0 < \varepsilon << 1$.

Since $C^\infty_c(I)$ is dense in $C(\bar{I})$ with respect to the sup-norm on I, (4) implies

$$\int_I f(x)\eta_\varepsilon(x)\,dx = 0 \quad \text{for any } \varepsilon > 0$$

Now let $\varepsilon \to 0$ and the $\delta \to 0$ we obtain $f(x_0) = 0$ for all Lebesgue points $x_0 \in I$. $\square$

We can now derive the optimality condition.

<u>Prop</u> (Euler equation) Let $u : I \to \mathbb{R}^d$ be a weak critical point of $\mathcal{E}$ and assume that $u \in C^2(I; \mathbb{R}^d)$ and $\mathcal{L} \in C^2(U)$ for some neighbourhood U of $\{x \mid (x, u(x), u'(x))\}$. Then u satisfies

$$\frac{d}{dx} \mathcal{L}_p(x, u(x), u'(x)) + \mathcal{L}_z(x, u(x), u'(x)) = 0 \quad \text{on } I$$

<u>proof</u> By (3) we have

$$\int_I [\mathcal{L}_z(x, u(x), u'(x)) \cdot \varphi + \mathcal{L}_p(x, u(x), u'(x)) \cdot \varphi'] \, dx = 0 \quad \forall \varphi \in C^\infty_c(I)$$

Integrating by part we get

$$\int_I [\mathcal{L}_z(x, u(x), u'(x)) + \frac{d}{dx} \mathcal{L}_p(x, u(x), u'(x))] \cdot \varphi(x) \, dx = 0$$

Choose $\varphi = (\varphi_1, \dots, \varphi_d)$ as $\varphi_k = 0$ if $k \neq i$ and
$\varphi_i = \eta \in C^\infty_c(I)$ for a fixed i.

Then we obtain

$$\int_I [\mathcal{L}_{z_i}(x,u,u') + \frac{d}{dx}\mathcal{L}_{p_i}(x,u,u')]\,\eta\, dx = 0$$

By one of the previous lemma we obtain

$$\mathcal{L}_{z_i}(x,u(x),u'(x)) - \frac{d}{dx}\mathcal{L}_{p_i}(x,u(x),u'(x)) = 0 \quad \text{on } I$$

In particular this holds for any $i = 1, \dots, d$; this concludes the proof $\square$

<u>Example</u> Let $V \in C^1(\mathbb{R}^3)$ and m a positive constant and consider the Lagrangian $\mathcal{L}(x,v) = \frac{m}{2}|v|^2 + V(x)$
where we have replaced z by and p by x and v respectively, while the independent variable x will be denoted by t.
We look for functions $x(t)$ which make the variational integrals

$$\mathcal{E}(x) = \int_{t_1}^{t_2} [\frac{m}{2}|\dot{x}|^2 + V(x)]\, dt$$

stationary. In mechanics $\mathcal{E}$ is the <u>action</u> of a motion $x(t)$ of a point mass m in a <u>conservative</u> force field $-V_x$ with the <u>potential energy</u> V and $T = \frac{1}{2}m(\dot{x})^2$ is the <u>kinetic energy</u> of the motion x. The <u>Euler equation</u> is equivalent to <u>Newton's</u> equation

$$m\ddot{x} = -V_x(x)$$

~~<u>Lemma</u> (Du Bois-Reymond) Assume $f \in C(I;\mathbb{R})$ satisfying~~

$$\cancel{\int_I f(x)v(x)\,dx = 0 \qquad \forall v \in C^\infty_c(I) \text{ s.t. } \int_I v(x)\,dx = 0}$$

~~Then f is constant in I~~

<u>prop</u> Suppose that $\mathcal{E}(u) \le \mathcal{E}(v)$ for all $v \in C^1(\bar{I}; \mathbb{R}^d)$ satisfying $u = v$ on ∂I and $\|u - v\|_{C^0} \le \delta$ for some suff. small $\delta > 0$. Moreover let $u \in C^2(I; \mathbb{R}^d) \cap C^1(\bar{I}; \mathbb{R}^d)$ and $\mathcal{L} \in C^2$. Then u satisfies the Euler equation.

<u>def</u> Any solution of $u \in C^2(I; \mathbb{R}^d)$ of Euler equation is said to be an extremal of $\mathcal{E}$.

<u>Rmk</u> any weak $\mathcal{E}$-minimizers in the sense of the above prop are $\mathcal{E}$-weak extremals if they are of <u>class C^2</u> and if the Lagrangian is of <u>class C^2</u>.

<u>Note</u> that a weak minimizer of $\mathcal{E}$ in C^1 <u>need not be</u> of class C^2.

<u>Example</u> $\mathcal{L}(x, z, p) = z^2(2x - p)^2$ $d = 1$, $I = (-1, 1)$

$$\mathcal{E}(u) = \int_{-1}^{1} u^2 (2x - u')^2 dx$$

the function $u(x) = \begin{cases} 0 & x \in [-1, 0] \\ x^2 & x \in [0, 1] \end{cases}$ is $C^1(\bar{I}; \mathbb{R})$

and the unique minimizer of $\mathcal{E}$ among all $v \in C^1(\bar{I}; \mathbb{R})$ s.t. $v(-1) = v(1) = 0$ but $u \notin C^2$

Although a weak C^1-extr. of $\mathcal{E}$ need not be of class C^2 it will nevertheless turn out that the Euler eq. holds true even if $\mathcal{L}_p$ is assumed to be continuous.

<u>Lemma</u> (DuBois-Reymond) Suff. $f \in L^1(I)$ satisfies

$$\int f(x) \eta'(x) dx = 0 \qquad \forall \eta \in C_c^\infty(I)$$

Then there is a constant $c \in \mathbb{R}$ s.t. $f = c$ a.e. on I.

proof Fix two Leb. point x_0 and $\xi \in I$ of f and set $c = f(x_0)$.

Supp. that $x_0 < \xi$ and $(x_0 - \varepsilon, \xi + \varepsilon) \subset\subset I$, $\varepsilon > 0$.

Chose a piecewise linear function $w \in C_c^0(I)$ s.t.

$$w(x) = \begin{cases} 1 & [x_0, \xi] \\ 0 & x \notin [x_0-\varepsilon, \xi+\varepsilon] \\ \varepsilon^{-1}(x - x_0 + \varepsilon) & [x_0-\varepsilon, x_0] \\ \varepsilon^{-1}(\xi - x + \varepsilon) & [\xi, \xi+\varepsilon] \end{cases}$$

By appx argument we infer that

$$\int f(x)\, w'(x) = 0$$

and it is equivaleent to

$$\frac{1}{\varepsilon}\int_{x_0-\varepsilon}^{x_0} f(x)\,dx - \frac{1}{\varepsilon}\int_{\xi}^{\xi+\varepsilon} f(x)\,dx = 0$$

letting $\varepsilon \to 0$ we arrive that $f(x_0) = f(\xi) = c$ for any Leb. point $\xi > x_0$.

By reversing the role of ξ and x_0 we get the result. $\square$

prop Let $u \in C^1(\bar{I}; \mathbb{R}^d)$ be an extremal of $\mathcal{E}$ s.t.

$$\int [L_z(x,u,u') \cdot \varphi + L_p(x,u,u') \cdot \varphi']\, dx = 0$$

$\forall \varphi \in C_c^\infty$. Then $\exists\, c \in \mathbb{R}^d$ s.t.

$$L_p(x, u(x), u'(x)) = c + \int_a^x L_z(s, u(s), u'(s))\, ds$$

$\forall x \in (a,b) = I$.

proof An integration by parts leads

$$\int_a^b L_z(x,u,u') \cdot \varphi = -\int_a^b \left(\int_a^x L_z(s, u(s), u'(s))\, ds \right) \cdot \varphi'$$

And we obtain the result by applying DuBois-Raymond's Lemma. $\square$

Note that the function

$$f(x) = c + \int_a^x L_z(s, u(s), u'(s))\, ds$$

is of class $C^1(\bar{I}; \mathbb{R}^d)$. Hence also $L_p(\cdot, u, u') \in C^1(\bar{I}; \mathbb{R}^d)$

and we drawn the Euler equation by differentiating w.r.t. x.

<u>Corollary</u> Any weak C^1-extremal of $\mathcal{E}$ satisfies

$$\frac{d}{dx} L_p(x,u(x),u'(x)) - L_z(x,u(x),u'(x)) = 0$$

<u>Rmk</u> Not allowed to apply the chain rule

By essentially the same reasoning we obtain.

<u>prop</u> If $u \in Lip(I;\mathbb{R}^d)$ is a weak-lip.-extremal of $\mathcal{E}$ then $\exists c \in \mathbb{R}^d$ such that

$$L_p(x,u,u') = c + \int_a^x L_z(s,u(s),u'(s))\,ds$$

holds true for almost all $x \in I$ and that

$$\frac{d}{dx} L_p + L_z = 0 \quad \text{a.e. on } I$$

Consider now the pb. $\inf_{\mathcal{C}} \mathcal{E}(u)$

$\mathcal{C} = C^1([a,b])$

We want now to derive another necessary condition the so called "natural boundary conditions". Let introduce the Euler operator $L_{\mathcal{E}}(u)$ by

$$L_{\mathcal{E}}(u) := L_z(\cdot,u,u') - \frac{d}{dx} L_p(\cdot,u,u')$$

Consider now $u \in C^2(\bar{I};\mathbb{R}^d)$, $L \in C^1(u)$ and $(\alpha,\beta) \subset I$. Then for all $\varphi \in C^1(\bar{I};\mathbb{R}^d)$ we get

$$\int_\alpha^\beta [L_z(\cdot,u,u')\cdot\varphi + L_p(\cdot,u,u')\cdot\varphi']\,dx =$$

$$= \int_\alpha^\beta L_{\mathcal{E}}(u)\cdot\varphi + [L_p(x,u(x),u'(x))\cdot\varphi(x)]_\alpha^\beta$$

The identity leads to the following result:

<u>prop</u> Suppose that $u \in C^1(\bar{I};\mathbb{R}^d)$, $L \in C^1(u)$ and

$$\delta\mathcal{E}(u,\varphi) = 0 \qquad \forall \varphi \in C^1(\bar{I};\mathbb{R}^d).$$

Then $L_p(x,u(x),u'(x))$ vanishes at the end points a and b of I.

<u>proof</u> (i) Assume that $u \in C^2(\bar{I};\mathbb{R}^d)$ and $L \in C^2(u)$. Clearly $\delta\mathcal{E}(u,\varphi) = 0$ holds for all $\varphi \in C_c^\infty$ whence $L_{\mathcal{E}}(u) = 0$

Then we have $[L_p \cdot \varphi]_a^b = 0$ if we choose $\alpha = a$ and $\beta = b$.

For any $\xi \in \mathbb{R}^d$ we can find $\varphi \in C^1(\bar{I};\mathbb{R}^d)$ s.t. $\varphi(a) = 0$ and $\varphi(b) = \xi$. Then we have that the function $f(x) = \mathcal{L}_p(x, u(x), u'(x))$ $x \in I$ satisfies

$f(b) \cdot \xi = 0$ $\forall \xi \in \mathbb{R}^d$ whence $f(b) = 0$ and similarly for $f(a)$.

Hence the claim is proved when $u \in C^2$ and $\mathcal{L} \in C^2$

(ii) Now only assume that $u \in C^1$ and $\mathcal{L} \in C^1$. By one of the above prop. we have that $\mathcal{L}_p$ is of class C^1 and so we still deduce that

$$\int [\mathcal{L}_z \cdot \varphi + \mathcal{L}_p \cdot \varphi'] = -\int L_{\mathcal{E}}(u) \cdot \varphi + [\mathcal{L}_p \cdot \varphi]_a^b$$ still holds

Then the previous argument still applies. $\square$

If $d = 1$ we can reduce $L_{\mathcal{E}}(u) = 0$ to a scalar equation of first order. If $\mathcal{L}(z, p)$ does not depend on x an important <u>first integral</u> is given by

$$\phi(z, p) = p \cdot \mathcal{L}_p(z, p) - \mathcal{L}(z, p)$$

$\searrow$ functions which are constant on the 1-graph of solution of Euler equation.

<u>prop</u> If $\mathcal{L}_x = 0$ then $\Phi(u(x), u'(x)) = \text{const.}$ on $\bar{I}$ for any solution $u \in C^2(\bar{I}; \mathbb{R}^d)$ of $L_{\mathcal{E}}(u) = 0$. (For <u>any</u> extremal of $\mathcal{E}$)

<u>proof</u>

$$\frac{d}{dx} \phi(u, u') = u' \cdot L_{\mathcal{E}}(u)$$

<u>Rmk</u> The <u>conservation law</u> $\Phi(u, u') = \text{Const}$ may have many solutions more then $L_{\mathcal{E}}(u) = 0$. For instance if $u(x) = \text{const} = c$ and $\mathcal{L}(c, 0) = -h$ then u is a solution of $\Phi(u, u') = h$ but it is a solution of $L_{\mathcal{E}}(u) = 0$ only if $\mathcal{L}(c, 0) = 0$

Thus one has to check that by solving $\Phi(u, u') = h$ we pick up a solution to $L_{\mathcal{E}}(u) = 0$.

Example $\quad \mathcal{L}(x,v) = \frac{m}{2}|v|^2 - V(x)$ and let

~~$E(x,p) = p\cdot L_p(x,p) - L(x,p)$~~

$$E(x,v) = v\cdot \mathcal{L}_v(x,v) - \mathcal{L}(x,v)$$

It is easy to see that $E(x,v) = \frac{m}{2}|v|^2 + V(x)$

$E(x,\dot{x})$ is the total energy of a motion $x(t)$ and $E(x(t),\dot{x}(t)) = \text{const.}$ for any extremal of

$$\mathcal{E}(x) = \int_{t_0}^{t_1} [\tfrac{m}{2}|\dot{x}|^2 - V(x)]\,dt.$$

$\Rightarrow$ Total Energy is conserved along any solution of Newton Eq.!

We can show that a rather large class of functions $u: [a,b] \to \mathbb{R}^d$ satisfies the conservation law $\Phi(u,u') = \text{const.}$ on $[a,b]$

Consider a function $u \in C^1(\bar{I};\mathbb{R})$ and a mapping $(t,\varepsilon) \mapsto x = \gamma(t,\varepsilon)$ of class C^1 on $\bar{I}\times(-\varepsilon_0,\varepsilon_0)$ $\varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (-\varepsilon_0,\varepsilon_0) =: I_0$ the mapping $\gamma(\cdot,\varepsilon)$ is C^1-diff. of $\bar{I}$ onto itself satisfying $\gamma(a,\varepsilon) = a$ and $\gamma(b,\varepsilon) = b$.

Furthermore assume $\frac{\partial}{\partial\varepsilon}\gamma(\cdot,\varepsilon) \in C^1(\bar{I})$ for $\varepsilon \in I_0$ and $\gamma(t,0) = t$ for all $t \in \bar{I}$. We call $\{\gamma(\cdot,\varepsilon)\}$ and admissible parameter variation and the family

$$v(t,\varepsilon) := u(\gamma(t,\varepsilon))$$

an admissible inner variation of u. Note that $v(t,0) = u(t)$

and $v(t,\varepsilon)$ has the same values as $u(t)$ at $t = a$ and b.

We set $\quad \lambda(t) = \frac{\partial\gamma}{\partial\varepsilon}(t,0)$ and

$$\psi(\varepsilon) = \int \mathcal{L}(t, v(t,\varepsilon), \dot{v}'(t,\varepsilon))\,dt = \mathcal{E}(v(t,\varepsilon))$$

$\dot{}$ is the derivative with respect to ~~time~~ t. If u is a minimizer in a class $\mathcal{C}$ of functions which is invariant with respect to ~~an in~~ a parameter variation $u \in \mathcal{C}$ then $v = u\circ\gamma \in \mathcal{C}$ then we have

$$\psi(\varepsilon) \geq \psi(0) \quad \text{for} \quad |\varepsilon| < \varepsilon_0$$

and therefore $\quad \psi'(0) = 0$

We set $\quad \delta \mathcal{E}(u,\lambda) = \psi'(0)$

and call $\delta\mathcal{E}(u,\lambda)$ the <u>first inner variation</u> of $\mathcal{E}$ at u in the direction λ

<u>prop</u> For $u \in C^1(\bar{I};\mathbb{R}^d)$ the inner variation of $\mathcal{E}$ in the direction λ is given by

$$\delta \mathcal{E}(u,\lambda) = \int_a^b [u' \cdot L_p(x,u,u') - L(x,u,u')]\lambda' - L_x \cdot \lambda) \, dx$$

Let $\lambda(x)$ be an arbitrary $C_c^\infty(I)$ and set

$$\tau(x,\varepsilon) = x - \varepsilon\lambda(x)$$

By taking $\sigma(t,\varepsilon) = \tau^{-1}(\cdot,\varepsilon)$ (Exercise) we define an admissible parameter variation. Hence.

<u>prop</u> if $\quad \frac{\partial}{\partial\varepsilon}\mathcal{E}(\sigma(\cdot,\varepsilon))\Big|_{\varepsilon=0} = 0 \quad (*)$ for any admissible inner variation $\sigma(t,\varepsilon)$ of u then we have

$$\delta\mathcal{E}(u,\lambda) = 0 \quad \text{for any all } \lambda \in C_c^\infty(I)$$

Then we have the following generalized conservation law.

<u>prop</u> if $(*)$ holds for any admissible inner variation then $\exists\, c$ such that the Erdmann's equation.

$$(\square)\ \Phi(x,u,u') = c - \int_a^x L_x(s,u(s),u'(s))\, ds \qquad \forall x \in I$$

holds true with $\Phi(x,z,p) = L(x,z,p) - p \cdot L_p(x,z,p)$. Moreover since $u \in C^1(\bar{I};\mathbb{R}^d)$ it follows that $L_x(\cdot,u,u')$ is $C(\bar{I})$. Then eq $(\square)$ implies that $\Phi(\cdot,u,u')$ is $C^1(\bar{I})$ and diff. w.r.t. x yields the Noether's equation

$$\frac{d}{dx}\Phi(x,u(x),u'(x)) + L_x(x,u(x),u'(x)) = 0 \qquad \text{in } I.$$

In particular if $L(z,p)$ we get the conservation law.

We want now to check that it is a minimum.

Take $\tilde{u} = u + \varphi$ then

$$\mathcal{E}(\tilde{u}) = \int_\Omega (u' + \varphi')^2 = \underbrace{\int (u')^2}_{\mathcal{E}(u)} + 2\int u'\varphi' + \int (\varphi')^2$$

$$= \mathcal{E}(u) + 2\underbrace{\int (u'')\varphi}_{=0} + \int (\varphi')^2 \geq \mathcal{E}(u)$$

use that $u \in C^2$

with equality if and only if $\int (\varphi')^2 = 0 \iff \varphi'(x) = 0 \iff \varphi(x) = 0$

1.2.2. <u>BOUNDARY CONDITIONS</u> (some examples)

Take $\mathcal{E}(u) = \int_a^b (u'^2 + u^2 + 7u)\,dx$

<u>Example 1</u>

$$X_1 = \{ u \in C^1([a,b]) \mid u(a) = A,\ u(b) = B \}$$
$$V_1 = \{ v \in C^1([a,b]) \mid v(a) = v(b) = 0 \}$$

$$\inf \{ \mathcal{E}(u) \mid u \in X_1 \}$$

$$\Phi(t) = \mathcal{E}(u + t\varphi) = \mathcal{E}(u) + t\int_a^b (2u'\varphi' + 2u\varphi + 7\varphi) + t^2 \int (\varphi'^2 + \varphi^2)$$
$\varphi \in V_1$

$$\Phi'(0) = \int (2u'\varphi' + 2u\varphi + \varphi)\,dx$$

If u is a critical point then $\underbrace{\int \underbrace{2u'}_{L_p}\varphi' + \underbrace{(2u+7)}_{L_u}\varphi = 0}_{\text{Euler equation}} \quad \forall \varphi \in V_1$

Integrating by parts we obtain $\int_a^b (-2u'' + 2u + 7)\varphi\,dx = 0$

and by one of the previous lemma we get

$$2u'' = 2u + 7$$

$$\Rightarrow \begin{cases} 2u'' = 2u + 7 \\ u(a) = A \\ u(b) = B \end{cases} \Big\} \text{ Dirichlet Boundary conditions}$$

<u>Example</u> $X_2 = \{ u \in \mathcal{C}^1([a,b]) \mid u(a) = A \}$

$V_2 = \{ v \in \mathcal{C}^1([a,b]) \mid v(a) = 0 \}$

As before we compute $\Phi(t)$ and $\Phi'(0)$ and we find

$$\int 2u'\varphi' + 2u\varphi + 7\varphi = 0 \qquad \forall \varphi \in V_2$$

What happens if we integrate by parts?

$$\int (-2u'' + 2u + 7)\varphi \, dx + [2\varphi u']_a^b = 0$$

$$\boxed{2u'(b)\varphi(b) + \int (-2u'' + 2u + 7) = 0} \quad \forall \varphi \in V_2 \qquad (*)$$

→ Limit ourselves to $\varphi \in \mathcal{C}_c^\infty(a,b) \subseteq V_2$ we get

$$\int (-2u'' + 2u + 7)\varphi = 0$$

$\underset{\text{lemme F.}}{\Rightarrow} \; -2u''(x) + 2u(x) + 7 = 0 \quad \forall x \in I$

→ Go back to $(*)$ we have $u'(b)\varphi(b) = 0 \quad \forall \varphi \in V_2$

which means that $u'(b) = 0$

We ended up with

$$\begin{cases} -2u'' + 2u + 7 = 0 & \\ u(a) = 0 & \text{DBC} \\ u'(b) = 0 & \underline{\text{Neumann BC}} \end{cases}$$

1.2.3 RECAP ON EULER LAGRANGE EQUATIONS

So far we have the following forms of ELE.

- $\delta\mathcal{E}(u,\varphi) = \boxed{\int [\mathcal{L}_z(x,u,u')\varphi + \mathcal{L}_p(x,u,u')\varphi']\,dx = 0}$

  this is the First integral form of ELE.

  Sufficient conditions:

  - $\mathcal{L}$ is of class C^1
  - u is of class C^1

- Assuming that φ vanishes at the boundaries If we integrate by part we obtain as before:

$$\boxed{\int [\mathcal{L}_z(x,u,u') + \frac{d}{dx}\mathcal{L}_p(x,u,u')]\varphi\,dx = 0}$$

this is the Second integral form of ELE Sufficient conditions

- $\mathcal{L}$ is of class C^2
- u is of class C^2

Exploiting FLCV we finally obtain

$$\boxed{\frac{d}{dx}\mathcal{L}_p(x,u,u') = \mathcal{L}_z(x,u,u')}$$

ELE in standard differential form.

Rmk $\mathcal{L}_{px}(x,u,u') + \mathcal{L}_{pz}(x,u,u')u' + \mathcal{L}_{pp}(x,u,u')u'' = \mathcal{L}_z(x,u,u')$

- Neumann condition in the general setting $I = (a,b)$

$$\int \mathcal{L}_z(x,u,u')\varphi + \mathcal{L}_p(x,u,u')\varphi'\,dx = 0$$

$$\Rightarrow \int \mathcal{L}_z(x,u,u')\varphi + [\mathcal{L}_p(x,u,u')]'\varphi + [\mathcal{L}_p(x,u,u')\varphi]_a^b$$

NEUMANN CONDITION : $\mathcal{L}_p(x,u,u') = 0$ in the endpoint

DU BOIS REYMOND ELE

$$\int_a^b \mathcal{L}_z(x,u,u')\varphi + \mathcal{L}_p(x,u,u')\varphi' = 0$$

$\hookrightarrow$ we want to transform it in φ'

Let us introduce the antiderivative of $\mathcal{L}_z(x,u,u')$, that is

$$\hat{\mathcal{L}}(x) := \int_a^x \mathcal{L}_z(y,u(y),u'(y))\,dy \quad \forall x \in [a,b]$$

Integrating by parts we find (assume $\varphi = 0$ at the boundary)

$$\int_a^b -\hat{\mathcal{L}}(x)\varphi' + \mathcal{L}_p(x,u,u')\varphi' = 0 = \int_a^b [-\hat{\mathcal{L}}(x) + \mathcal{L}_p(x,u,u')]\varphi'$$

where now φ' "is any C^∞ function with 0 average" since $\int \varphi'(x)dx = \varphi(b) - \varphi(a) = 0$

DBR Lemma $\Rightarrow$ $-\hat{\mathcal{L}}(x) + \mathcal{L}_p(x,u,u') = \text{constant}$

$$\Rightarrow \quad \boxed{\mathcal{L}_p(x,u,u') = \text{const} + \int_a^x \mathcal{L}_z(y,u(y),u'(y))\,dy} \quad \text{1st order ODE}$$

ELE in DBR form does not require any second derivative (just u and $\mathcal{L}$ of class C^1)

ELE in ERDMANN FORM

$\mathcal{L}$ does not depend on x

$$\boxed{\mathcal{L}_p(u,u')u' - \mathcal{L}(u,u') = \text{const}}$$

Rmk Standard ELE and Erdmann forms are not equivalent

(1) If u satisfies ELE standard then u satisfies ELE-ERDMANN

(2) If u satisfies ELE-ERDMANN then u satisfies ELE standard for all point s.t. $x \in [a,b]$ s.t. $\dot{u}(x) \neq 0$

· How decide whether an extremal is a minimizer?

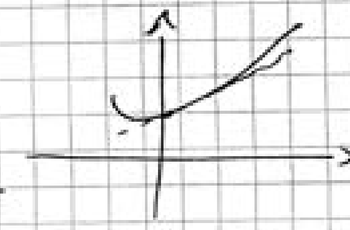
There are several other necessary conditions to be satisfied by minimizers for instance a minimizer must satisfy the necessary Legendre condition.

$$\mathcal{L}_{p^i p^k}(x, u(x), u'(x))\xi^i \xi^k \geq 0 \quad \forall \xi \in \mathbb{R}^d \ \forall x \in \bar{I}.$$

In fact if $u \in C^1(\bar{I}; \mathbb{R}^d)$ is a weak minimizer of $\mathcal{E}$ and $\mathcal{L}$ is C^2 then the function $\Phi(\varepsilon) = \mathcal{E}(u + \varepsilon\varphi)$ satisfies $\Phi(0) \leq \Phi(\varepsilon)$ whence $\Phi'(0) = 0$ and $\Phi''(0) \geq 0$. The second relation ~~leads to~~ yields to

$$\underbrace{\delta^2 \mathcal{E}(u, \varphi)}_{\text{the second variation}} \geq 0 \quad \forall \varphi \in C_c^\infty(I; \mathbb{R}^d)$$

MINIMALITY THROUGH CONVEXITY : A ~~FIRST~~ GLIMPSE



- Basic fact on convex functions

If ~~this~~ $f : \mathbb{R} \to \mathbb{R}$ is convex then $\exists m \in \mathbb{R}$ s.t.

$$f(x) \geq f(x_0) + m(x - x_0)$$

proof -> trivial when $f \in C^2$

this is true actually in any dimension $(m \in \mathbb{R}^d)$

Consider now the functional

$$\mathcal{E}(u) = \int_I \mathcal{L}(x, u(x), u'(x))\, dx$$

~~assume~~ $\mathcal{L} \in C^1$ subject to BC ~~u(a) and u(b)~~ $u(a) = A$, $u(b) = B$

Assume u^* is a solution to ELE with BC

prop (informal)
If $(z, p) \mapsto \mathcal{L}(x, z, p)$ is convex w.r.t. (z, p) for every $x \in [a, b]$

Then u^* is a minimum point for $\mathcal{E}(u)$!

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<u>example</u>

let $\mathcal{E}(u) = \int [(u')^2 + q(x)u^2] dx$ the functional

which has to be minimized among all functions of class $C^1(\bar{I})$ s.t. $u(a) = u(b) = 0$ as well as the constraint

$$\int u^2 dx = 1$$

then any minimizer u with $u \in C^2$ satisfies.

$$\begin{cases} -u'' + q(x)u = \lambda u \\ u(a) = u(b) = 0 \end{cases}$$

for some λ. λ is an eigenvalue of the Sturm-Liouville operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

for zero bc. conditions.

proof Let us take any $w \in C^1$ with same BC.

$w = u_0 + v$, $v \in C^1$ with 0 BC.

$$\mathcal{E}(w) = \mathcal{E}(u_0+v) = \int_a^b \mathcal{L}(x, u_0+v, u_0'+v')\,dx$$

$$\underset{\text{by convexity}}{\geq} \int_a^b \mathcal{L}_z(x,u_0,u_0')v + \mathcal{L}_p(x,u_0,u_0')v' = 0$$

$\Rightarrow$ $\mathcal{E}(w) \geq \mathcal{E}(u_0)$ for any admissible w Example $\rightarrow$

Second tool to prove minimality

Trivial lemma

prop Let $\mathcal{F}: \mathcal{C} \to \mathbb{R}$ and $\mathcal{H}: \mathcal{C} \to \mathbb{R}$ be two real valued functionals on a set $\mathcal{C}$ and suppose that for some element $u_0 \in \mathcal{C}$ the following conditions are satisfied

(i) $\mathcal{F}(u_0) = \mathcal{H}(u_0)$ and $\mathcal{F}(u) \geq \mathcal{H}(u)$ $\forall u \in \mathcal{C}$

(ii) $\mathcal{H}(u) \geq \mathcal{H}(u_0)$ $\forall u \in \mathcal{C}$

Then we have $\mathcal{F}(u) \geq \mathcal{F}(u_0)$ $\forall u \in \mathcal{C}$

proof Since $\mathcal{F}(u_0) = \mathcal{H}(u_0)$ we can write

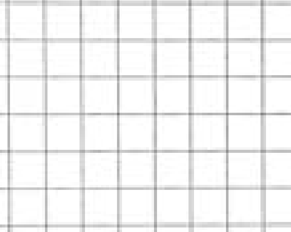
$$\mathcal{F}(u) - \mathcal{F}(u_0) = \mathcal{F}(u) - \mathcal{H}(u_0)$$
$$= \underbrace{[\mathcal{F}(u) - \mathcal{H}(u)]}_{\geq 0} + \underbrace{[\mathcal{H}(u) - \mathcal{H}(u_0)]}_{\geq 0}$$

By (i) and (ii) we get $\mathcal{F}(u) \geq \mathcal{F}(u_0)$ $\forall u \in \mathcal{C}$

Example

$$\min \left\{ \int_0^1 (\dot{u}^2 - 1)^2 dx \;\middle|\; u \in C^1 \;\; u(0) = 2,\; u(1) = 4 \right\}$$

$\mathcal{L}(p) = (p^2-1)^2$ $\rightsquigarrow$ DOUBLE WELL potential!



claim $u_0(x) = 2 + 2x$ is the unique minimizer

Let $\mathcal{E}(u)$ the given functional and

and $\mathcal{M}(u) = \int_0^1 \hat{\mathcal{L}}(u')\,dx$

where $\hat{\mathcal{L}}$ is the convexification of $\mathcal{L}$

$\mathcal{E}(u) \geq \mathcal{M}(u)$ $\qquad \left(\mathcal{L}(p) \geq \hat{\mathcal{L}}(p) \quad \forall\, p \in \mathbb{R} \right)$

$\mathcal{E}(u_0) = \mathcal{M}(u_0)$ $\qquad \left(\mathcal{L}(z) = \hat{\mathcal{L}}(z) \right)$

u_0 is a minimum point of $\mathcal{M}$ (because $\hat{\mathcal{L}}$ is convex)

Lagrange Multipliers

Finally we want to state the Lagrange multipliers theorem for variational problems with the so called isoperimetric side conditions. These are constraints of the form

$$\mathcal{G}(u) = \text{const} \quad (*)$$

where $\mathcal{G}$ is a given functional of the form

$$\mathcal{G}(u) = \int_a^b G(x, u(x), u'(x))\, dx$$

with a Lagrangian $G(x,z,p)$ of class C^1 in a neighbourhood of 1-graph u. We are looking for functions $u : \bar{I} \to \mathbb{R}^d$, $I = (a,b)$ which minimize a given variational integral $\mathcal{E}(u)$ among all C^1 mappings $\bar{I} \to \mathbb{R}^d$ satisfying boundary conditions as well as condition $(*)$.

<u>prop</u> Suppose that u is a weak minimizer of the variational integral $\mathcal{E}$ in the class $\mathcal{C}$ of all functions $v \in C^1(\bar{I}; \mathbb{R})$ satisfying the boundary conditions $v(a) = \alpha$, $v(b) = \beta$ and the constraint $\mathcal{G}(v) = c \in \mathbb{R}$. Assume also $\delta\mathcal{G}(u,\varphi)$ does not vanish for all $\varphi \in C_c^\infty(I)$. Then there is a real number λ such that

$$\delta\mathcal{E}(u,\varphi) + \lambda\,\mathcal{G}(u,\varphi) = 0 \quad \forall \varphi \in C_c^\infty(I;\mathbb{R}^d).$$

Moreover if $u \in C^2(I;\mathbb{R}^d)$ then

$$\frac{d}{dx} H_p(x,u,u') - H_z(x,u,u') = 0$$

where $H := F + \lambda G$.

<u>proof</u> By assumption $\exists\, \psi \in C_c^\infty(I;\mathbb{R}^d)$ s.t. $\delta\mathcal{G}(u,\psi) = 1$.

Using a $\varphi \in C_c^\infty(I;\mathbb{R}^d)$ we define $\Phi : Q \to \mathbb{R}$ and $\Psi : Q \to \mathbb{R}$ on $Q := \{(\varepsilon,t) \mid |\varepsilon| < \varepsilon_0, |t| < t_0\}$, $\varepsilon_0 > 0$, $t_0 \ll 1$ by

$$\Phi(\varepsilon,t) := \mathcal{E}(u + \varepsilon\varphi + t\psi), \quad \Psi(\varepsilon,t) = \mathcal{G}(u + \varepsilon\varphi + t\psi)$$

Since $\Psi_t(0,0) = 1$ we can apply the implicit function theorem and for $|\varepsilon| \ll 1$ we obtain a function $\tau \in C^1(-\varepsilon, \varepsilon_0)$ with $\tau(0) = 0$ s.t. $(\varepsilon, \tau(\varepsilon)) \in Q$ for $|\varepsilon| < \varepsilon_0$ and

$$\Psi(\varepsilon, \tau(\varepsilon)) = c \quad \text{for all} \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0)$$

whence

$$\tau'(0) = -\Psi_\varepsilon(0,0)$$

Furthermore the functions u and $v = u + \varepsilon\varphi + t\psi$ satisfy the same boundary condition at $x = a$ and $x = b$ and their C^1-distance

$$\|v - u\|_{C^1(I)} \leq |\varepsilon| \, \|\varphi\|_{C^1} + |t| \leq \|\psi\|_{C^1}$$

tends to 0 as $\varepsilon \to 0$ and $t \to 0$ thus we have

$$\Phi(\varepsilon, \tau(\varepsilon)) \geq \Phi(0,0) \quad \text{for} \quad |\varepsilon| \ll 1$$

and it follows that

$$\Phi_\varepsilon(0,0) + \Phi_t(0,0)\,\tau'(0) = 0$$

By introducing the Lagrange multipliers λ as

$$\lambda := -\Phi_t(0,0) = -\delta\mathcal{G}(u,\psi)$$

which is independent of φ we arrive at the equation

$$\Phi_\varepsilon(0,0) + \lambda\,\Psi_\varepsilon(0,0) = 0$$

thus we have $\quad \delta\mathcal{E}(u,\varphi) + \lambda\,\delta\mathcal{G}(u,\varphi) \quad \forall \varphi \in C_c^\infty(I; \mathbb{R}^d)$