Chapter 2
Calculus of Variations in 1D and regularity

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Contents

1 Examples of Existence and Non-Existence

We will see in this section how to prove the existence for a very standard example of variational problem, the technique being useful to understand how to attack more general problems. We will consider a functional involving first derivatives of the unknown u, but the existence of the minimizers will not be proven in the space C^1. We will instead set the problem in a Sobolev space, since Sobolev spaces have better compactness properties and their norm is more adapted to study variational problems.

1.1 1D examples

Let us firstly by introducing some examples in one dimension.

The geodesic problem Given two points in a given space X we want to find the minimal-length connecting them. The length of a curve $\omega : [0,1] \to \mathbb{R}^d$ is defined as

$$\ell(\omega) := \sup \left\{ \sum_{i=0}^{n-1} |\omega(t_k) - \omega(t_{k-1})| \mid n \geqslant 1,\ 0 = t_0 = \cdots = t_n = 1 \right\}.$$

Notice that of $\omega \in C^1$ then we have that

$$\ell(\omega) = \int_0^1 |\dot{\omega}(t)|\, dt.$$

So given two points x_0 and x_1, the geodesic problem takes the following form

$$\inf\{\ell(\omega) \mid \omega \in AC([0,1];X),\ \omega(0) = x_0, \omega(1) = x_1\},$$

where $AC([0,1];X)$ stands for absolutely continuous curves defined on $[0,1]$ and valued in X.

The Ramsey model for the optimal groth of an economic activity In this model we describe the financial situation of an economic activity (say, a small firm owned by an individual) by a unique number representing its capital, and aggregating all relevant information such as money in the bank account, equipment, properties, workers,etc. This number can evolve in time and will be denoted by $k(t)$. The evolution of k depends on how much the firm produces and how much the owner decides to "consume". We denote by $f(k)$ the production when the capital level is k, and we usually assume that f is increasing and often concave. We also assume that capital depreciates at a fixed rate $\delta > 0$ and we call $c(t)$ the consumption at time t. We then have $k'(t) = f(k(t)) - \delta k(t) - c(t)$. The goal of the owner is to optimize the consumption bearing in mind that reducing the consumption for some time allows the capital to be kept high, letting it grow even further, to finally be consumed later. More precisely, the owner has a utility function $U : \mathbb{R} \to \mathbb{R}$, also increasing and concave, and a discount rate $r > 0$, and his goal is o optimize

$$\int_0^T e^{-rt} U(c(t))\, dt,$$

where T is fixed time horizon. One can take $T = \infty$ or $T < \infty$ and possibly add a final pay-off $\psi(k(T))$. The maximization problem can be considered a maximization over c, and $k(T)$ can be deduced from c, or everything can expressed in terms of k. Positivity constraints are also reasonably added on c and on k. The problem becomes a classical calculus of variations problems of the form

$$\sup \left\{ \int_0^T e^{-rt} U(f(k) - \delta k - k')\, dt + \psi(k(T)) \,|\, k(0) = k_0,\ k \geqslant 0,\ c = f(k) - \delta k - k' \geqslant 0 \right\}.$$

1.2 Existence of Minimizers

We consider an interval $I = [a, b] \subset \mathbb{R}$ and a continuous function $F : I \times \mathbb{R} \to \mathbb{R}$ that we assume bounded from below.

Theorem 1.1. *Let us consider the problem:*

$$\boxed{\inf \left\{ \mathcal{E}(u) \mid u \in H^1(I), u(a) = A, u(b) = B \right\}.}$$

where

$$\mathcal{E}(u) := \int_a^b \left(F(x, u(x)) + |u'(x)|^2 \right) dx$$

This minimization problem admits a solution.

Proof. Take a minimizing sequence $\{u_n\}$ such that $\mathcal{E}(u_n) \to \inf \mathcal{E}$. The functional $\mathcal{E}$ is composed of two terms (the one with F and the one with $|u'|^2$), and their sum is bounded from above. Since they are both bounded from below, we can deduce that they are also both bounded from above. In particular, we obtain an upper bound for $\|u_n'\|_{L^2}$. Since the boundary values of u_n are fixed, applying the Poincaré inequality of Lemma 1.3 to the functions $x \mapsto u_n(x) - \frac{B-A}{b-a}(x - a) - A$, we obtain a bound on $\|u_n\|_{H^1}$ (i.e., the L^2 norms of u_n and not only of u_n' are bounded).

Hence, $(u_n)_n$ is a bounded sequence in H^1, and we can extract a subsequence which weakly converges in H^1 to a function u. In dimension one, the weak convergence in H^1 implies the uniform convergence, and in particular, the pointwise convergence on the boundary. We then deduce from $u_n(a) = A$ and $u_n(b) = B$ that we have $u(a) = A$ and $u(b) = B$, i.e., that u is an admissible competitor for our variational problem. We just need to show $\mathcal{E}(u) \leqslant \liminf \mathcal{E}(u_n) = \inf \mathcal{E}$ in order to deduce $\mathcal{E}(u) = \inf \mathcal{E}$ and the optimality of u. In this case, the uniform convergence $u_n \to u$ implies:

$$\int_a^b F(x, u_n(x))\, dx \to \int_a^b F(x, u(x))\, dx,$$

and proves the continuity of the first integral term.

We now observe that the map $H^1 \ni u \mapsto u' \in L^2$ is continuous, and hence the weak convergence of u_n to u in H^1 implies $u_n' \rightharpoonup u'$ in L^2. An important property of the weak convergence in any Banach space is the fact that the norm itself is lower semicontinuous, so that:

$$\|u'\|_{L^2} \leqslant \liminf \|u_n'\|_{L^2},$$

and hence:

$$\int_a^b |u'(x)|^2\, dx \leqslant \liminf \int_a^b |u_n'(x)|^2\, dx.$$

This concludes the proof. □

Remark 1.2 (The direct method of calculus of variations)**.** The strategy we have just seen is very general and typical in the calculus of variations. It is called the *direct method* and requires a topology (or a notion of convergence) to be found on the set of admissible competitors such that:

- There is compactness (any minimizing sequence admits a convergent subsequence, or at least a properly built minimizing sequence does so).
- The functional that we are minimizing is lower semicontinuous, i.e., $\mathcal{E}(u) \leqslant \liminf \mathcal{E}(u_n)$ whenever $u_n \to u$.

1.3 Non-Existence

We consider in this section a very classical example of a variational problem in 1D which has no solution. With this aim, let us consider:

$$\inf \left\{ \mathcal{E}(u) \mid u \in H_0^1([0,1]) \right\},$$

where

$$\boxed{\mathcal{E}(u) := \int_0^1 \left(\left| |u'(x)|^2 - 1 \right| + |u(x)|^2 \right) dx.}$$

It is clear that for any admissible u, we have $\mathcal{E}(u) > 0$: indeed, $\mathcal{E}$ is composed of two non-negative terms, and they cannot both vanish for the same function u. In order to have $\int_0^1 |u(x)|^2 \, dx = 0$, one would need $u = 0$ constantly, but in this case, we would have $u' = 0$ and $\mathcal{E}(u) = 1$. On the other hand, we will prove $\inf \mathcal{E} = 0$, which shows that the **minimum cannot be attained**.

To do so, we consider the following sequence of Lipschitz functions u_n: we first define $U : [0,1] \to \mathbb{R}$ as:

$$U(x) = \frac{1}{2} - |x - \frac{1}{2}|.$$

It satisfies $U(0) = U(1) = 0$, $|U| \leqslant \frac{1}{2}$, and $|U'| = 1$ a.e. (U is Lipschitz continuous but not C^1). We then extend U as a 1-periodic function on $\mathbb{R}$, that we call $\tilde{U}$, and then set $u_n(x) = \frac{1}{n}\tilde{U}(nt)$. The function u_n is $\frac{1}{n}$-periodic, satisfies $u_n(0) = u_n(1) = 0$ again, and $|u_n'| = 1$ a.e. We also have $|u_n| \leqslant \frac{1}{2n}$. If we compute $\mathcal{E}(u_n)$, we easily see that:

$$\mathcal{E}(u_n) \leqslant \frac{1}{4n^2} \to 0,$$

which shows $\inf \mathcal{E} = 0$. The above example is very useful to understand the relation between compactness and semicontinuity. Indeed, the sequence u_n which we built is such that u_n converges uniformly to 0, and u_n' is bounded in L^∞. This means that we also have $u_n' \overset{*}{\rightharpoonup} 0$ in L^∞ (indeed, a sequence which is bounded in L^∞ admits a weakly-* convergent subsequence and the limit in the sense of distributions of u_n' can only be the derivative of the limit of u_n, i.e., 0). This means that, if we use weak convergence in Sobolev spaces (weak convergence in L^∞ implies weak convergence in L^2, for instance), then we have compactness. Yet, the limit is the function $u = 0$, but we have $\mathcal{E}(u) = 1$ while $\lim \mathcal{E}(u_n) = 0$, which means that semicontinuity fails. This is due to the lack of convexity of the double-well function $W(p) = \left| |p|^2 - 1 \right|$: indeed, the 0 derivative of the limit function u is approximated through weak convergence as a limit of a rapidly oscillating sequence of functions u_n' taking values ± 1, and the values of W at ± 1 are better than the value at 0 (which would not have been the case if W was convex). On the other hand, it would have been possible to choose a stronger notion of convergence, for instance, strong H^1 convergence. In this case, if we have $u_n \to u$ in H^1, we deduce $u_n' \to u'$ in L^2, and $\mathcal{E}(u_n) \to \mathcal{E}(u)$. We would even obtain continuity (and not just semicontinuity) of $\mathcal{E}$. Yet, what would be lacking in this case is the compactness of minimizing sequences (and the above sequence u_n, which is a minimizing sequence since $\mathcal{E}(u_n) \to 0 = \inf \mathcal{E}$, proves that it is not possible to extract strongly convergent subsequences). This is hence a clear example of the difficult task of choosing a suitable convergence for applying the direct method of calculus of variations: not too strong, otherwise there is no convergence; not too weak, otherwise lower semicontinuity could fail.

We finish this section by observing that the problem of non-existence of the minimizer of $\mathcal{E}$ does not depend on the choice of the functional space. Indeed, even if we considered:

$$\inf \left\{ \mathcal{E}(u) := \int_0^1 \left(\left| |u'(x)|^2 - 1 \right| + |u(x)|^2 \right) dx : u \in C^1([0,1]), u(0) = u(1) = 0 \right\},$$

we would have $\inf \mathcal{E} = 0$ and $\mathcal{E}(u) > 0$ for every competitor, i.e., no existence. To show this, it is enough to modify the above example in order to produce a sequence of C^1 functions. In this case, it will not be possible to make the term $\int_0^1 \left| |u'(x)|^2 - 1 \right| dx$ exactly vanish, since this requires $u' = \pm 1$, but for a C^1 function, this means either $u' = 1$ everywhere or $u' = -1$ everywhere, and neither choice is compatible with the boundary data. On the other hand, we can fix $\delta > 0$, use again the Lipschitz function U introduced above, and define a function $U_\delta : [0,1] \to \mathbb{R}$ such that:

$$U_\delta = U \text{ on } \left[\delta, \frac{1}{2} - \delta\right] \cup \left[\frac{1}{2} + \delta, 1 - \delta\right], \quad U_\delta(0) = U_\delta(1) = 0, \quad |U_\delta| \leqslant \frac{1}{2}, \quad |U_\delta'| \leqslant 2.$$

We then extend U_δ to a 1-periodic function $\tilde{U}_\delta$ defined on $\mathbb{R}$, and set $u_{n,\delta}(x) := \frac{1}{n}\tilde{U}_\delta(nt)$. We observe that we have $|u_{n,\delta}'| \leqslant 2$ and $|u_{n,\delta}'| = 1$ on a set:

$$A_{n,\delta} = \bigcup_{k=0}^{n-1} \left(\left[\frac{k}{n} + \frac{\delta}{n}, \frac{2k+1}{2n} - \frac{\delta}{n}\right] \cup \left[\frac{2k+1}{2n} + \frac{\delta}{n}, \frac{k+1}{n} - \frac{\delta}{n}\right] \right),$$

whose measure is $1 - 4\delta$. We then have:

$$\mathcal{E}(u_{n,\delta}) = \int_{[0,1]\setminus A_{n,\delta}} \left| |u_{n,\delta}'(x)|^2 - 1 \right| dx + \int_0^1 |u_{n,\delta}(x)|^2 \, dx \leqslant 12\delta + \frac{1}{4n^2}.$$

This shows $\inf \mathcal{E} \leqslant 12\delta$ and, $\delta > 0$ being arbitrary, $\inf \mathcal{E} = 0$.

2 Optimality Conditions

We consider here the necessary optimality conditions for a typical variational problem. The result will be presented in 1D, but we will see that the procedure is exactly the same in higher dimensions. We consider a function $\mathcal{L} : [a,b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. We assume that $\mathcal{L}$ is C^1 in (x,p) for a.e. $x \in [a,b]$.

2.1 The Euler-Lagrange Equation

We start from the minimization problem:

$$\min\left\{\mathcal{E}(u) := \int_a^b \mathcal{L}(x, u(x), u'(x))\, dx : u \in \mathcal{U}, u(a) = A, u(b) = B\right\},$$

where $\mathcal{U}$ is a functional space which could be, in most cases, a Sobolev space. We assume anyway $\mathcal{U} \subset W^{1,1}([a,b];\mathbb{R}^d)$, which guarantees both the existence of a suitably defined derivative and the continuity of all functions in $\mathcal{U}$. We also assume that for every $u \in \mathcal{U}$, the negative part of $\mathcal{L}(\cdot, u, u')$ is integrable, so that $\mathcal{E}$ is a well-defined functional from $\mathcal{U}$ to $\mathbb{R}\cup\{+\infty\}$. We assume that u is a solution of such a minimization problem and that $u + C_c^\infty((a,b)) \subset \mathcal{U}$. This means that for every $\varphi \in C_c^\infty((a,b))$, we have $\mathcal{E}(u) \leqslant \mathcal{E}(u + t\varphi)$ for small t. We now fix the minimizer u and a perturbation φ, and consider the one-variable function:

$$g(t) := \mathcal{E}(u + t\varphi)$$

which is defined in a neighborhood of $t = 0$ and minimal at $t = 0$. We now want to compute $g'(0)$.

In order to do so, we will assume that for every $u \in \mathcal{U}$ with $\mathcal{E}(u) < +\infty$, there exists a $\delta > 0$ such that we have the following integrability condition:

$$x \mapsto \sup_{v\in B(u(x),\delta), p\in B(u'(x),\delta)} \{|\nabla_u \mathcal{L}(x, v, p)| + |\nabla_p \mathcal{L}(x, v, p)|\} \in L^1([a, b]).$$

We will discuss later some sufficient conditions on $\mathcal{L}$ that guarantee this is satisfied. If this is the case, then we can differentiate with respect to t the function $t \mapsto \mathcal{L}(x, u + t\varphi, u' + t\varphi')$, and obtain:

$$\frac{d}{dt}\mathcal{L}(x, u+t\varphi, u'+t\varphi') = \nabla_u \mathcal{L}(x, u+t\varphi, u'+t\varphi')\cdot\varphi + \nabla_p \mathcal{L}(x, u+t\varphi, u'+t\varphi')\cdot\varphi'.$$

Since we assume $\varphi \in C_c^\infty((a,b))$, both φ and φ' are bounded, so that for small t, we have $(|\varphi(x)|, |\varphi'(x)|) \leqslant \delta$, and we can apply the assumption to obtain domination in L^1 of the pointwise derivatives. This shows that, for small t, we have:

$$g'(t) = \int_a^b (\nabla_u \mathcal{L}(x, u + t\varphi, u' + t\varphi')\cdot\varphi + \nabla_p \mathcal{L}(x, u + t\varphi, u' + t\varphi')\cdot\varphi')\, dx.$$

In particular, we have:

$$g'(0) = \int_a^b (\nabla_u \mathcal{L}(x, u, u')\cdot\varphi + \nabla_p \mathcal{L}(x, u, u')\cdot\varphi')\, dx.$$

Imposing $g'(0) = 0$, which comes from the optimality of u, means precisely that we have, in the sense of distributions, the following differential equation, known as the *Euler-Lagrange equation*:

$$\frac{d}{dx}(\nabla_p \mathcal{L}(x, u, u')) = \nabla_u \mathcal{L}(x, u, u').$$

This is a second-order differential equation (on the right-hand side, we have the derivative in x of a term already involving u'), and as such, requires the choice of two boundary conditions. These conditions are $u(a) = A$ and $u(b) = B$, which means that we are not facing a Cauchy problem (where we would prescribe $u(a)$ and $u'(a)$).

2.2 Energy Conservation

When speaking of variational principles in mathematical physics, the word *energy* has multiple meanings due to an ambiguity in the physical language, which is increased by the mathematicians' use of the notion. Sometimes we say that the motion should minimize the total energy (and we think in this case of an integral in x of a cost involving kinetic energy and potential energy; a more adapted name which can be found in the literature is *action*), while on other occasions, we can say that the energy is preserved along the evolution (and in this case, we think of a quantity computed at every x).

Mathematically, this can be clarified in the following way: assume the integrand in a variational problem is independent of the first variable and of the form $\mathcal{L}(x, u, p) = \frac{1}{2}|p|^2 + V(u)$. The Euler-Lagrange equation of the corresponding minimization problem would be $u'' = \nabla_u V(u)$. If we take the scalar product with u', we obtain:

$$\frac{d}{dx}\left(\frac{1}{2}|u'(x)|^2\right) = u'(x) \cdot u''(x) = u'(x) \cdot \nabla_u V(u(x)) = \frac{d}{dx}V(u(x)).$$

This shows that the difference $\frac{1}{2}|u'|^2 - V(u)$ is constant in x. We see that the minimization of the integral of $\frac{1}{2}|u'|^2 + V(u)$ (i.e., the sum of the kinetic energy and of V) implies that the difference $\frac{1}{2}|u'|^2 - V(u)$ is constant. Which quantity should be called *energy* is then a matter of convention (or taste).

In the very particular case $V = 0$, this result reads as "minimizers of the integral of the square of the speed have constant speed," and is a well-known fact for geodesics .

Remark 2.1 (Beltrami formula)**.** The energy conservation principle described above is a particular case of the so-called *Beltrami formula*, which is valid whenever the integrand $\mathcal{L}$ does not depend explicitly on x. In this case, from the Euler-Lagrange equation:

$$\frac{d}{dx}\left(\nabla_p \mathcal{L}(u, u')\right) = \nabla_u \mathcal{L}(u, u'),$$

we can deduce:

$$\mathcal{L}(u, u') - u' \cdot \nabla_p \mathcal{L}(u, u') = \text{constant}.$$

2.3 Examples of Growth Conditions on $\mathcal{L}$

We now discuss under which conditions on $\mathcal{L}$ and $\mathcal{U}$ we can guarantee the integrability condition:

$$x \mapsto \sup_{v \in B(u(x),\delta), p \in B(u'(x),\delta)} \{|\nabla_u \mathcal{L}(x, v, p)| + |\nabla_p \mathcal{L}(x, v, p)|\} \in L^1([a, b]).$$

We will not give an exhaustive classification of all possible cases, which is probably impossible to do. We will only consider three examples.

1. Case of Polynomial Growth: $\mathcal{L}$ has growth of order α in terms of p and is of the form:

$$\mathcal{L}(x,u,p) = c|p|^\alpha + F(x,u),$$

which is the example we considered for existence in the case $\alpha = 2$. Note that in this case, a natural choice for the space $\mathcal{U}$ is $\mathcal{U} = W^{1,\alpha}([a,b])$, since any minimizing sequence for $\mathcal{E}$ will be bounded in $\mathcal{U}$, and the arguments of Section 1.2 prove that a minimizer exists.

3. Case of Multiplicative Growth: $\mathcal{L}$ has growth of order α in terms of p, but it has a multiplicative form:

$$\mathcal{L}(x,u,p) = a(x,u)|p|^\alpha,$$

for $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ bounded from below and above by positive constants. In this case, we will also choose $\mathcal{U} = W^{1,\alpha}([a,b])$, since minimizing sequences will also be bounded. However, we observe that Section 1.2 does not yet provide a proof of existence since the semi-continuity of the functional $u \mapsto \int \mathcal{L}(x,u(x),u'(x))dx$ still has to be proven.

3. Non-Standard Growth Example: $\mathcal{L}$ is of the form:

$$\mathcal{L}(x,u,p) = e^{h(p)} + F(x,u),$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a C^1, Lipschitz continuous, and convex function such that $\lim_{|p|\to\infty} h(p) = +\infty$.

Analysis of the Three Cases:

1. In the first case, let us assume that F is C^1 with respect to x and that $\nabla_u F$ is continuous in (x,u). In this case, we have:

   $$\nabla_u \mathcal{L}(x,u,p) = \nabla_u F(x,u), \quad \nabla_p \mathcal{L}(x,u,p) = \alpha c|p|^{\alpha-2}p.$$

   For every $u \in \mathcal{U}$, we have $u \in C^0$ and $u' \in L^\alpha$. In particular, assuming $|u| \leqslant M$, we have:

   $$\sup\{|\nabla_u F(x,v)| : x \in [a,b], |v| \leqslant M + \delta\} < +\infty.$$

   Concerning $\nabla_p \mathcal{L}$, we have:

   $$\sup_{v \in B(u(x),\delta), p \in B(u'(x),\delta)} |\nabla_p \mathcal{L}(x,v,p)| \leqslant C(|u'(x)|^{\alpha-1} + \delta^{\alpha-1}),$$

   and this is integrable in x as soon as $u' \in L^{\alpha-1}$, which is satisfied since we even have $u' \in L^\alpha$.

2. In the second case, we assume that a is C^1 in x and that $\nabla_u a$ is continuous in (x, u). We have:

$$\nabla_u \mathcal{L}(x, u, p) = \nabla_u a(x, u)|p|^\alpha, \quad \nabla_p \mathcal{L}(x, u, p) = \alpha a(x, u)|p|^{\alpha-2}\alpha.$$

   Using $u \in C^0$ and assuming $|u| \leqslant M$, we have:

$$\sup_{x\in[a,b],\ |v|\leqslant M+\delta} \{|\nabla_u a(x, u)|\} (|u'(x)| + \delta)^\alpha \leqslant C(|u'(x)|^\alpha + \delta^\alpha).$$

   This is integrable in x as soon as $u' \in L^\alpha$, which is true for every $u \in \mathcal{U}$.

3. The third case is trickier because of the non-standard growth of $\mathcal{L}$ in p. A natural choice for the space $\mathcal{U}$ would be a Sobolev-Orlicz space, which imposes integrability of $e^{|u'|}$, but it is not necessary to do so. Indeed, one can take, for arbitrary p, $\mathcal{U} = W^{1,p}([a, b])$. Thanks to the growth of $\mathcal{L}$, any minimizing sequence will be bounded in such a space, and the semicontinuity results will also allow us to prove the existence of a solution.

2.4 Transversality Conditions

We consider now the case where the Dirichlet boundary conditions on $u(a)$ and $u(b)$ are replaced by penalizations on the values of u at $t = a, b$. Let us consider for instance the problem

$$\min\{\mathcal{E}(u) + \psi_0(u(a)) + \psi_1(u(b)) \mid u \in \mathcal{U}\}, \tag{2.1}$$

where, again, we set

$$\mathcal{E}(u) := \int_a^b \mathcal{L}(t, u(t), u'(t))\, dt. \tag{2.2}$$

We assume that u is a minimizer and we set $u_t := u + t\varphi$, but we do not assume $\varphi \in C_c^\infty((a, b))$, since we are no longer obliged to preserve the values at the boundary points. Let us set $\mathcal{E}_\psi(u) := \mathcal{E}(u) + \psi_0(u(a)) + \psi_1(u(b))$ and $j_\psi(t) := \mathcal{E}_\psi(u_t)$. The optimality of u provides $f_\psi(0) \leqslant f_\psi(t)$ and we want to differentiate f_ψ in terms of t.

The computation is exactly the same as before (and requires the very same assumptions) concerning the term $f(\varepsilon) = \mathcal{E}(u_t)$ and is very easy for the boundary terms. We then obtain

$$f'_\psi(0) = \int_a^b \left(\nabla_x \mathcal{L}(t, u, u') \cdot \varphi + \nabla_v \mathcal{L}(t, u, u') \cdot \varphi'\right) dt + \nabla\psi_0(u(a))\cdot\varphi(a) + \nabla\psi_1(u(b))\cdot\varphi(b). \tag{2.3}$$

This derivative should vanish for arbitrary φ, and it is possible to first consider $\varphi \in C_c^\infty((a, b))$. In this case we obtain exactly as before the Euler–Lagrange equation

$$\frac{d}{dt}\left(\nabla_v \mathcal{L}(t, u, u')\right) = \nabla_x \mathcal{L}(t, u, u'). \tag{2.4}$$

We now assume that u and $\mathcal{L}$ are such that $t \mapsto \nabla_x \mathcal{L}(t, u(t), u'(t))$ is L^1. This implies that $t \mapsto \nabla_v \mathcal{L}(t, u(t), u'(t))$ is $W^{1,1}$ and in particular continuous. Moreover, the term $\int_a^b \nabla_v \mathcal{L}(t, u, u') \cdot \varphi' \, dt$ can be integrated by parts, thus obtaining

$$\begin{aligned} j'_\psi(0) = & \int_a^b \nabla_x \mathcal{L}(t, u, u') \cdot \varphi \, dt - \int_a^b \left(\frac{d}{dt} \nabla_v \mathcal{L}(t, u, u') \right) \cdot \varphi \, dt \\ & + (\nabla \psi_0(u(a)) - \nabla_v \mathcal{L}(a, u(a), u'(a))) \cdot \varphi(a) \\ & + (\nabla \psi_1(u(b)) + \nabla_v \mathcal{L}(b, u(b), u'(b))) \cdot \varphi(b). \end{aligned}$$

Since the first two integrals coincide thanks to the Euler–Lagrange equation, we are finally only left with the boundary terms. Their sum should vanish for arbitrary φ, which provides

$$\nabla \psi_0(u(a)) - \nabla_v \mathcal{L}(a, u(a), u'(a)) = 0, \tag{2.5}$$
$$\nabla \psi_1(u(b)) + \nabla_v \mathcal{L}(b, u(b), u'(b)) = 0. \tag{2.6}$$

These two boundary conditions, called transversality conditions, replace in this case the Dirichlet boundary conditions on $u(a)$ and $u(b)$, which are no longer available, and allow us to complete the equation. Of course, it is possible to combine the problem with fixed endpoints and the Bolza problem with penalization on the boundary, fixing one endpoint and penalizing the other. The four possible cases, with their Euler–Lagrange systems, are the following.

1. For the problem

$$\min \left\{ \int_a^b \mathcal{L}(t, u(t), u'(t)) \, dt : u(a) = A, u(b) = B \right\}, \tag{2.7}$$

the Euler–Lagrange system is:

$$\frac{d}{dt} (\nabla_v \mathcal{L}(t, u, u')) = \nabla_x \mathcal{L}(t, u, u') \quad \text{in } (a, b), \tag{2.8}$$
$$u(a) = A, \tag{2.9}$$
$$u(b) = B. \tag{2.10}$$

2. For the problem

$$\min \left\{ \int_a^b \mathcal{L}(t, u(t), u'(t)) \, dt + \psi_0(u(a)) : u(b) = B \right\}, \tag{2.11}$$

the Euler–Lagrange system is:

$$\frac{d}{dt} (\nabla_v \mathcal{L}(t, u, u')) = \nabla_x \mathcal{L}(t, u, u') \quad \text{in } (a, b), \tag{2.12}$$
$$\nabla_v \mathcal{L}(a, u(a), u'(a)) = \nabla \psi_0(u(a)), \tag{2.13}$$
$$u(b) = B. \tag{2.14}$$

3. For the problem

$$\min\left\{\int_a^b \mathcal{L}(t,u(t),u'(t))\,dt + \psi_1(u(b)) : u(a) = A\right\}, \tag{2.15}$$

the Euler–Lagrange system is:

$$\frac{d}{dt}\left(\nabla_v \mathcal{L}(t,u,u')\right) = \nabla_x \mathcal{L}(t,u,u') \quad \text{in } (a,b), \tag{2.16}$$

$$u(a) = A, \tag{2.17}$$

$$\nabla_v \mathcal{L}(b,u(b),u'(b)) = -\nabla\psi_1(u(b)). \tag{2.18}$$

4. For the problem

$$\min\left\{\int_a^b \mathcal{L}(t,u(t),u'(t))\,dt + \psi_0(u(a)) + \psi_1(u(b))\right\}, \tag{2.19}$$

the Euler–Lagrange system is:

$$\frac{d}{dt}\left(\nabla_v \mathcal{L}(t,u,u')\right) = \nabla_x \mathcal{L}(t,u,u') \quad \text{in } (a,b), \tag{2.20}$$

$$\nabla_v \mathcal{L}(a,u(a),u'(a)) = \nabla\psi_0(u(a)), \tag{2.21}$$

$$\nabla_v \mathcal{L}(b,u(b),u'(b)) = -\nabla\psi_1(u(b)). \tag{2.22}$$

3 Regularity and the Lavrientev phenomenon

Let us consider again the problem

$$\inf\{\mathcal{E}(u) \mid u \in \mathcal{U}\} = m \tag{3.23}$$

where $\mathcal{U} := \{u \in W^{1,\alpha}(a,b) \,|\, u(a) = A,\ u(b) = B\}$ and $\mathcal{E}(u) = \int_a^b \mathcal{L}(x,u,u')$ with $\mathcal{L}$ of class C^2.

Before focusing on some regularity issue for the 1 dimensional case, let us consider the following existence theorem without proving it (we will see it for the general case later). Assume that the following hypothesis are satisfied

(H1) there exist $\alpha > q \geqslant 1$ and $c_1 > 0$, $c_2, c_3 \in \mathbb{R}$ such that for every $(x,u,p) \in [a,b]\times\mathbb{R}\times\mathbb{R}$

$$\mathcal{L}(x,u,p) \geqslant c_1|p|^\alpha + c_2|u|^q + c_3,$$

we will see that this ensures existence (notice that this condition says that the Lagrangian has a polynomial growth).

(H2) for every $\delta > 0$ there exists $c(\delta)$ such that for every $(x,u,p) \in [a,b]\times[-\delta,\delta]\times\mathbb{R}$

$$|\mathcal{L}(x,u,p)|, |\nabla_u \mathcal{L}(x,u,p)|, |\nabla_p \mathcal{L}(x,u,p)| \leqslant c(\delta)(1+|p|^\alpha),$$

this ensures that any minimizer of (3.23) satisfied the Euler-Lagrange equations we have studied above.

(H3) for all $(x,u,p) \in [a,b] \times [-\delta,\delta] \times \mathbb{R}$

$$D^2_{pp}\mathcal{L}(x,u,p) > 0.$$

Theorem 3.1 (Existence). *Let $\mathcal{L} \in C^2$ and satisfy (H1) and (H3). Assume that there exists u_0 such that $\mathcal{E}(u_0) < \infty$ then there exists a unique minimizer to* (3.23)

Remark 3.2. (i) Notice that for the existence the hypothesis (H3) can be replace by asking only convexity in the variable p.

(ii) The theorem easily applies to the case of Dirichlet energy $\mathcal{L}(x,u,p) = \frac{1}{2}|p|^2$ with $\alpha = 2$. And also tot he natural generalization

$$\mathcal{L}(x,u,p) = \frac{1}{\alpha}|p|^\alpha + F(x,u)$$

where F is continuous and bounded from below (as we have seen at the beginning of this long lecture)

3.1 Regularity

Lemma 3.3. *Let $\mathcal{L} \in C^2$ and satisfy (H1),(H2) and (H3). Then any minimizer Then any minimizer $\overline{u} \in W^{1,\alpha}(a,b)$ of* (3.23) *is in fact in $W^{1,\infty}(a,b)$, and the Euler-Lagrange equation holds almost everywhere, i.e.,*

$$\frac{d}{dx}\left[\nabla_p \mathcal{L}(x,\overline{u},\overline{u}')\right] = \nabla_u \mathcal{L}(x,\overline{u},\overline{u}'), \quad a.e.\ x \in (a,b).$$

Proof. First, we know that the following equation holds:

$$\int_a^b \left[\nabla_u \mathcal{L}(x,\overline{u},\overline{u}')v + \nabla_p \mathcal{L}(x,\overline{u},\overline{u}')v'\right] dx = 0, \quad \forall v \in C_0^\infty(a,b).$$

We then divide the proof into two steps.

Step 1. Define:

$$\varphi(x) := \nabla_p \mathcal{L}(x,\overline{u}(x),\overline{u}'(x)) \quad \text{and} \quad \psi(x) := \nabla \mathcal{L}(x,\overline{u}(x),\overline{u}'(x)). \tag{3.24}$$

We easily see that $\varphi \in W^{1,1}(a,b)$ and that $\varphi'(x) = \psi(x)$ for almost every $x \in (a,b)$, which means that

$$\frac{d}{dx}\left[\nabla_p \mathcal{L}(x,\overline{u},\overline{u}')\right] = \nabla_u \mathcal{L}u(x,\overline{u},\overline{u}'), \quad \text{a.e. } x \in (a,b). \tag{3.25}$$

Indeed, since $\overline{u} \in W^{1,\alpha}(a,b)$, and hence $\overline{u} \in L^\infty(a,b)$, we deduce from (H2) that $\psi \in L^1(a,b)$. We also have from (3.24) that

$$\int_a^b \psi(x)v(x)\,dx = -\int_a^b \varphi(x)v'(x)\,dx, \quad \forall v \in C_0^\infty(a,b).$$

Since $\varphi \in L^1(a,b)$ (from (H2)), we have by the definition of weak derivatives the claim, namely $\varphi \in W^{1,1}(a,b)$ and $\varphi' = \psi$ a.e.

Step 2. Since $\varphi \in W^{1,1}(a,b)$, we have that $\varphi \in C^0([a,b])$, which means that there exists a constant $c_5 > 0$ such that:

$$|\varphi(x)| = |\nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x))| \leqslant c_5, \quad \forall x \in [a,b]. \tag{3.26}$$

Since $\overline{u}$ is bounded (and even continuous), let us say $|\overline{u}(x)| \leqslant \delta$ for every $x \in [a,b]$, we have from (H3) (notice that this hypothesis implies convexity of the lagrangian in p) that:

$$\mathcal{L}(x,u,0) \geqslant \mathcal{L}(x,u,p) - p\nabla_p \mathcal{L}(x,u,p), \quad \forall (x,u,p) \in [a,b] \times [-\delta,\delta] \times \mathbb{R}.$$

Combining this inequality with (H1), we find that there exists $c_6 \in \mathbb{R}$ such that, for every $(x,u,p) \in [a,b] \times [-\delta,\delta] \times \mathbb{R}$,

$$p\nabla_p \mathcal{L}(x,u,p) \geqslant \mathcal{L}(x,u,p) - \mathcal{L}(x,u,0) \geqslant c_1|p|^\alpha + c_6.$$

Using (3.26) and the above inequality, we find:

$$c_1|\overline{u}'|^\alpha + c_6 \leqslant \overline{u}'\nabla_p \mathcal{L}(x,\overline{u},\overline{u}') \leqslant |\overline{u}'||\nabla_p \mathcal{L}(x,\overline{u},\overline{u}')| \leqslant c_5|\overline{u}'|, \quad \text{a.e. } x \in (a,b),$$

which implies, since $\alpha > 1$, that $|\overline{u}'|$ is uniformly bounded. Thus, the lemma. □

Theorem 3.4. *Let $\mathcal{L} \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$ satisfy (H1), (H2), and (H3). Then any minimizer of* (3.23) *is in $C^\infty([a,b])$.*

Proof. We divide the proof into two steps.
Step 1. We know from Lemma 3.3 that

$$x \mapsto \varphi(x) := \nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x))$$

is in $W^{1,1}(a,b)$ and hence it is continuous. Consider now the Legendre transform of the function $\mathcal{L}$, that is

$$\mathcal{L}^*(x,u,v) := \sup_{\xi \in \mathbb{R}} \{vp - \mathcal{L}(x,u,\xi)\},$$

then $\mathcal{L}^* \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$ (we will show it in 2 lectures!) and, for every $x \in [a,b]$, we have

$$\varphi(x) = \nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x)) \iff \overline{u}'(x) = \mathcal{L}^*_v(x, u(x), \varphi(x)).$$

Since $\nabla_v \mathcal{L}^*$, $\overline{u}$, and φ are continuous, we infer that $\overline{u}'$ is continuous and hence $\overline{u} \in C^1([a,b])$. We therefore deduce that $x \mapsto \nabla_u \mathcal{L}(x, u(x), u'(x))$ is continuous, which, combined with the fact that

$$\frac{d}{dx}[\varphi(x)] = \nabla_u \mathcal{L}(x, u(x), u'(x)), \quad \text{a.e. } x \in (a,b),$$

(or equivalently ,by properties of $\mathcal{L}^*$, $\varphi' = -\nabla_u \mathcal{L}^*(x,u,\varphi)$) leads to $\varphi \in C^1([a,b])$.

Step 2. Considering now the system:

$$\begin{cases} \overline{u}'(x) = \nabla_v \mathcal{L}^*(x, \overline{u}(x), \varphi(x)), \\ \varphi'(x) = -\nabla_u \mathcal{L}^*(x, \overline{u}(x), \varphi(x)), \end{cases}$$

we can start our iteration. Indeed, since $\mathcal{L}^*$ is C^∞ and $\overline{u}$ and φ are C^1, we deduce from our system that, in fact, $\overline{u}$ and φ are C^2. Returning to the system, we get that $\overline{u}$ and φ are C^3. Finally, we conclude that $\overline{u}$ is C^∞, as desired. □

3.2 The Lavrientev phenomenon

We have seen here that, under some assumptions on the growth of the Lagrangian function, we are able to prove existence results as well as the well-posedness of the Euler-Lagrangian equations for a "weak" minimizer (remember that we are working on Sobolev spaces). So it is quite natural to have the impression that we have found the right space to work with and the correct "generalization" of minimum problems involving an integral energy whose Lagrangian has a polynomial (superlinear indeed) growth. Unfortunately this is just an impression (as it is often the case in math!). If we want to consider the minimizer of problem (3.23) as a "generalized solution" of the problem *Minimize* $\mathcal{E}(u)$ *in the class of smooth functions with* $u(a) = A$ *and* $u(b) = B$ we should at least expect that the infimum of (3.23) agrees with the infimum on the class of smooth functions, i.e.

$$\inf_{u\in W^{1,1},\, u(a)=A,\, u(b)=B} \mathcal{E}(u) = \inf_{u\,\text{smooth},\, u(a)=A,\, u(b)=B} \mathcal{E}(u).$$

Theorem 3.5 (Mania's example)**.** *Let*

$$\mathcal{L}(x,u,p) := (x-u^3)^2 p^6, \quad \mathcal{E}(u) := \int_0^1 \mathcal{L}(x,u(x),u'(x))\,dx.$$

Let

$$\mathcal{W}_\infty := \{u \in W^{1,\infty}(0,1) : u(0) = 0,\ u(1) = 1\},$$
$$\mathcal{W}_1 := \{u \in W^{1,1}(0,1) : u(0) = 0,\ u(1) = 1\}.$$

Then

$$\inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_\infty\} > \inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_1\} = 0.$$

Moreover, $u(x) = x^{1/3}$ *is a minimizer of* $\mathcal{E}$ *over* $\mathcal{W}_1$*.*

Lemma 3.6. *Let* $0 < a < b < 1$ *and*

$$\mathcal{W}_{a,b} := \{u \in W^{1,\infty}(a,b) \,|\, u(a) = \frac{1}{4}a^{1/3},\ u(b) = \frac{1}{2}b^{1/3},\ \frac{1}{4}x^{1/3} \leqslant u(x) \leqslant \frac{1}{2}x^{1/3}\ \forall x \in [a,b]\}.$$

If $\mathcal{L}(x,u,p) = (x-u^3)^2 p^6$ *and*

$$\mathcal{E}_{a,b}(u) := \int_a^b \mathcal{L}(x,u(x),u'(x))\,dx,$$

then

$$\mathcal{E}_{a,b}(u) \geqslant c_0 b,$$

for every $u \in \mathcal{W}_{a,b}$ *and for* $c_0 = 7^2 3^5 2{-}185^{-5}$*.*

Proof. Theorem 3.5 **Step 1** We first prove that if $u \in \mathcal{W}_\infty$, then there exist $0 < a < b < 1$ such that $u \in \mathcal{W}_{a,b}$, namely

$$\begin{cases} u(a) = \frac{1}{4}a^{1/3}, \\ u(b) = \frac{1}{2}b^{1/3}, \\ \frac{1}{4}x^{1/3} \leqslant u(x) \leqslant \frac{1}{2}x^{1/3},\ \forall x \in [a,b]. \end{cases} \tag{3.27}$$

The existence of such a and b is easily seen . Let

$$A := \{a \in (0,1) \,|\, u(a) = \frac{1}{4}a^{1/3}\},$$

$$B := \{b \in (0,1) \,|\, u(b) = \frac{1}{2}b^{1/3}\}.$$

Since u is Lipschitz, $u(0) = 0$, and $u(1) = 1$, it follows that $A \neq \emptyset$ and $B \neq \emptyset$. Next, choose

$$a := \max\{\alpha : \alpha \in A\}, \quad \beta := \min\{\beta : \beta \in B, \beta > a\}.$$

It is then clear that a and v satisfy the required (3.27).

Step 2 We may therefore use the lemma to deduce that, for every $u \in \mathcal{W}_\infty$,

$$\mathcal{E}(u) = \int_0^1 (x - u^3)^2 u'^6 \, dx \geqslant \int_a^b (x - u^3)^2 u'^6 \, dx \geqslant c_0 b > c_0 > 0.$$

Thus,

$$\inf\{\mathcal{E}(u) : u \in \mathcal{W}_\infty\} \geqslant c_0 > 0.$$

Step 3 The fact that $u(x) = x^{1/3}$ is a minimizer of $\mathcal{E}$ over all $u \in \mathcal{W}_1$ is trivial. Hence,

$$\inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_1\} = 0.$$

This achieves the proof of the theorem.

□

4 Some reminders

4.1 Sobolev Spaces in 1D

Given an open interval $I \subset \mathbb{R}$ and an exponent $p \in [1, +\infty]$, we define the Sobolev space $W^{1,p}(I)$ as:

$$W^{1,p}(I) := \left\{ u \in L^p(I) : \exists g \in L^p(I) \text{ s.t. } \int u\varphi' \, dx = -\int g\varphi \, dx \text{ for all } \varphi \in C_c^\infty(I) \right\}.$$

The function g, if it exists, is unique, and will be denoted by u' since it plays the role of the derivative of u in the integration by parts.

The space $W^{1,p}$ is endowed with the norm:

$$\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|u'\|_{L^p}.$$

With this norm, $W^{1,p}$ is a Banach space, separable if $p < +\infty$, and reflexive if $p \in (1, \infty)$.

All functions $u \in W^{1,p}(I)$ admit a continuous representative, which moreover satisfies:

$$u(t_0) - u(t_1) = \int_{t_1}^{t_0} u'(x) \, dx.$$

This representative is differentiable a.e., and the pointwise derivative coincides with the function u' a.e. Moreover, for $p > 1$, the same representative is also Hölder continuous, of exponent $\alpha = 1 - \frac{1}{p} > 0$. The injection from $W^{1,p}$ into $C^0(I)$ is compact if I is bounded.

If $p = 2$, the space $W^{1,p}$ can be given a Hilbert structure, choosing as a norm:

$$\sqrt{\|u\|_{L^2}^2 + \|u'\|_{L^2}^2},$$

and is denoted by H^1.

Higher-order Sobolev spaces $W^{k,p}$ can also be defined for $k \in \mathbb{N}$ by induction as follows:

$$W^{k+1,p}(I) := \{u \in W^{k,p}(I) : u' \in W^{k,p}(I)\},$$

and the norm in $W^{k+1,p}$ is defined as $\|u\|_{W^{k,p}} + \|u'\|_{W^{k,p}}$. In the case $p = 2$, the Hilbert spaces $W^{k,2}$ are also denoted by H^k.

4.2 Hilbert Spaces

A Hilbert space is a Banach space whose norm is induced by a scalar product: $\|x\| = \sqrt{x \cdot x}$.

Theorem (Riesz): If H is a Hilbert space, for every $\xi \in H'$ there is a unique vector $h \in H$ such that $\langle \xi, x \rangle = h \cdot x$ for every $x \in H$, and the dual space H' is isomorphic to H.

In a Hilbert space H, we say that x_n weakly converges to x and we write $x_n \rightharpoonup x$ if $h \cdot x_n \to h \cdot x$ for every $h \in H$. Every weakly convergent sequence is bounded, and if $x_n \rightharpoonup x$, using $h = x$, we find:

$$\|x\|^2 = x \cdot x = \lim x \cdot x_n \leqslant \liminf \|x\| \|x_n\|,$$

i.e., $\|x\| \leqslant \liminf \|x_n\|$.

In a Hilbert space H, every bounded sequence x_n admits a weakly convergent subsequence.

4.3 Weierstrass Criterion for the Existence of Minimizers, Semicontinuity

The most common way to prove that a function admits a minimizer is called the *direct method in the calculus of variations*. It simply consists of the classical Weierstrass Theorem, possibly replacing continuity with semicontinuity.

Definition: On a metric space X, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous* (l.s.c. in short) if for every sequence $x_n \to x$, we have:

$$f(x) \leqslant \liminf f(x_n).$$

A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semicontinuous* (u.s.c. in short) if for every sequence $x_n \to x$, we have:

$$f(x) \geqslant \limsup f(x_n).$$

Definition: A metric space X is said to be *compact* if from any sequence x_n, we can extract a convergent subsequence $x_{n_k} \to x \in X$.
Theorem (Weierstrass): If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and X is compact, then there exists an $\overline{x} \in X$ such that:

$$f(\overline{x}) = \min\{f(x) : x \in X\}.$$

Proof: Define $\# := \inf\{f(x) : x \in X\} \in \mathbb{R} \cup \{-\infty\}$ ($\# = +\infty$ only if f is identically $+\infty$, but in this case, any point in X minimizes f). By definition, there exists a minimizing sequence x_n, i.e., points in X such that $f(x_n) \to \#$. By compactness, we can assume $x_n \to \overline{x}$. By lower semicontinuity, we have:

$$f(\overline{x}) \leqslant \liminf f(x_n) = \#.$$

On the other hand, we have $f(\overline{x}) \geqslant \#$ since $\#$ is the infimum. This proves $\# = f(\overline{x}) \in \mathbb{R}$ and this value is the minimum of f, realized at $\overline{x}$.