CHAPTER II: THE DIRECT METHOD (1d)

We will start the discussion on the existence via the direct methods for some simple 1D pb. we will se in the next lectures how to extend everything to the general setting.

First of all we will not look for C^1 solution but set the problem in a Sobolev space. Sobolev spaces enjoy better compactness property

(I) Reminder : Sobolev spaces and Weierstrass criterion

Given an open interval I and $p \in [1, +\infty]$ we define $W^{1,p}(I)$ as.

$$W^{1,p} := \left\{ u \in L^p(I) \mid \exists g \in L^p(I) \text{ s.t. } \int u \varphi' dx = -\int g \varphi \, dx \quad \forall \varphi \in C_c^\infty(I) \right\}$$

we will denote g by u'!

norm $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_p$ -> $W^{1,p}$ is separable ~~...~~ $p \neq +\infty$ reflexive if $p \in (1,\infty)$. All functions $u \in W^{1,p}(I)$ admits a continuous repr. $u(b) - u(a) = \int_a^b u'(x)\, dx$

This rep. is diff. a.e. and the pointwise der. coincides with u' a.e.

For $p > 1$ the same rep. is Hölder cont. $\alpha = 1 - \frac{1}{p}$. the injection from $W^{1,p}$ into $C^0(I)$ is always compact if I is bounded.

If $p = 2$ $W^{1,p}$ is an Hilbert and it is denoted by H^1

The most common way to prove that a function admits a minimizer is called THE DIRECT METHOD IN THE CALCULUS OF VARIATIONS. It simply consists of the classical Weierstrass thm.

<u>def</u> (l.s.c.) On a metric space X a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be l.s.c. if $\forall x_n \to x$ we have

$$f(x) \leq \liminf f(x_n)$$

<u>Thm</u> (Weierstrass) If $f : X \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and X is compact

Then $\exists \bar{x} \in X$ s.t. $f(\bar{x}) = \min \{ f(x) : x \in X \}$

Existence of minimizers

$I=[a,b]\subset\mathbb{R}$ and $F: I\times\mathbb{R}\to\mathbb{R}$ cont. and bounded from below. Denote by $\mathcal{E}(u)$ the following energy

$$\mathcal{E}(u)=\int_a^b F(x,u(x))+|u'(x)|^2\,dx$$

thm Consider the problem

$$\inf\{\mathcal{E}(u)\mid u\in H^1(I),\ u(a)=\alpha,\ u(b)=\beta\}<+\infty$$

Then it admits at least a solution.

proof

Take a minimizing sequence $\{u_n\}$ such that $\mathcal{E}(u_n)\to\inf\mathcal{E}(u)$.

The functional $\mathcal{E}$ is composed of two terms (the one with F and the one with $|u'|^2$) and their sum is bounded from above. Since they are both bounded from below we can deduce that they are also both bounded from above. In particular we obtain and upper bound for $\|u'\|_{L^2}$. Since the boundary values are fixed by Poincaré inequality we obtain a bound on $\|u\|_{H^1}$.

Hence $\{u_n\}$ is bounded in H^1 and we can extract a subsequence which weakly converges in H^1 to a function u.

In one dimension the weak convergence in H^1 implies the uniform convergence and in particular the pointwise convergence on the boundary. We then deduce from $u_n(a)=\alpha$ and $u_n(b)=\beta$ that we have $u(a)=\alpha$ and $u(b)=\beta$. This means that u is and admissible competitor for the variational pb.

We just need to show $\mathcal{E}(u)\le\liminf\mathcal{E}(u_n)$ to deduce optimality of u. In this case the uniform cv $u_n\to u$ implies

$$\int_a^b F(x,u_n(x))\to\int_a^b F(x,u(x))$$

Observe now that the map $H^1 \ni u \mapsto u' \in L^2$ is cont. and hence the weak convergence of u_n to u in H^1 implies $u_n' \rightharpoonup u'$ in L^2.
An important property of the weak cv in any Banach space is the fact that the norm itself is l.s.c. so that

$$\|u'\|_{L^2} \le \liminf \|u_n'\|_{L^2}$$

and hence $\int_a^b |u'|^2 \le \liminf \int |u_n'|^2$ $\square$

Rmk The strategy we have just seen is very general and typical in CALVA. It is called the direct method and requires a topology (or a notion of cv) to be found on the set of admissible competitors s.t.

(i) there is compactness (any minimizing sequence admits a convergent subsequence or at least a properly found minimizing sequence does so)

(ii) The functional that we are minimizing is l.s.c. whenever $u_n \to u$!

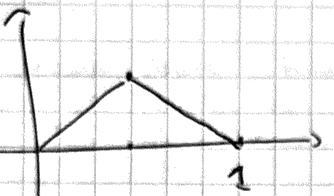
NON EXISTENCE RESULT

Let us consider $\inf \{ \mathcal{E}(u) \mid u \in H_0^1(0,1) \}$

where $\mathcal{E}(u) = \int_0^1 (\, | |u'|^2 - 1 | + |u|^2) \, dx$

It is clear that for any admissible u we have $\mathcal{E}(u) > 0$. Indeed $\mathcal{E}$ is composed of two non negative terms and they cannot vanish for the same u: $\int_0^1 u^2 dx = 0$ one would need $u = 0$ constantly but in this case we would have $u' = 0$ and $\mathcal{E}(u) = 1$.
On the other hand we will prove $\inf \mathcal{E} = 0$ which shows the the MINIMUM CANNOT BE ATTAINED.
Consider the following sequence of lip. functions u_n:
– First define

$$U : [0,1] \to \mathbb{R} \quad \text{s} \quad U(x) = \tfrac{1}{2} - |x - \tfrac{1}{2}|$$



$U(0)=U(1)=0$ $\quad |U|\le \frac{1}{2}$

and $\quad |U'|=1$ a.e.

We then extend U as a 1-periodic fct on $\mathbb{R}$ that we call $\tilde{U}$ and we set $u_n(x)=\frac{1}{n}\tilde{U}(nx)$. The function u_n is $\frac{1}{n}$-periodic

satisfies $u_n(0)=u_n(1)=0$ and $|u'|=1$ a.e.

We have ~~lse~~ ~~u_n is~~ $|u_n|\le \frac{1}{2n}$. If we compute $\mathcal{E}(u_n)$ we easily see that

$$\mathcal{E}(u_n)\le \frac{1}{4n^2}\to 0$$

The above example is very useful to understand the relation between compactness and semicontinuity! The sequence u_n is such that u_n cv. uniformly to 0 and u'_n is bounded in L^∞. This means that we also have $u'_n \overset{*}{\rightharpoonup} L^\infty$ (a sequence which is bounded in L^∞ ~~means~~ admits a w-* convergent subsequence and the limit in the sense of dist. u'_n can only be the derivative of the limit of u_n) This means that if we use weak convergence in Sobolev spaces (w cv in L^∞ implies w. cv in L^2) then we have compactness.

Yet the limit is the function $u=0$ BUT we have $\mathcal{E}(u)=1$ while $\lim \mathcal{E}(u_n)=0$ which means that semicontinuity fails!!

This is due to the LACK OF CONVEXITY of the double well fct. $W(p)=||p|^2-1|$: indeed the 0 derivative of the limit function u is approximated through weak cv as a limit of a rapidly oscillating sequence of functions u'_n taking values ± 1 and the value of W at ± 1 are better than at 0 (which would not have been the case if W was convex!) On the other hand it would have been possible to choose a stronger notion of convergence for instance strong H^1 cv. In this case we would even obtain continuity: $u_n\to u$ in $H^1 \Rightarrow u'_n\to u'$ in L^2 and $\mathcal{E}(u_n)\to\mathcal{E}(u)$!

Yet, what would be lacking in this case is the compactness of minimizing sequence! (the above sequence which is a minimizing one since $\mathcal{E}(u_n) \to 0$ proves that it is not possible to extract strongly convergent subsequences!) This is a clear example of the difficult task of choosing a suitable convergence for applying the direct method: not too strong otherwise there is no convergence, not too weak otherwise lower semicontinuity could FAIL!

We finish this section by observing that the problem of non-existence of the minimizer of $\mathcal{E}$ does not depend on the choice of the functional space. Consider

$$\inf\left\{ \mathcal{E}(u) := \int (||u'|^2 - 1| + |u|^2 \;\middle|\; u \in C^1([0,1]),\ u(0) = u(1) = 0 \right\}$$

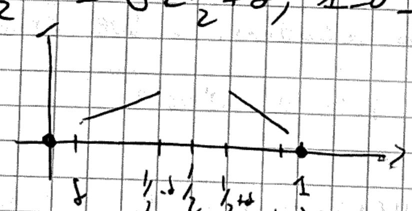
we would have $\inf \mathcal{E} = 0$ and $\mathcal{E}(u) > 0$ for every competitor.

To show this it is enough to modify the above example in order to produce a sequence of C^1 fct. In this case it will not possible to make the term $\int_0^1 ||u'|^2 - 1| \, dx$ exactly vanish since this requires $u' = \pm 1$ but for a C^1 fct this means either $u' = 1$ everywhere or $u' = -1$ and this is not compatible with the boundary data!

On the other hand we can fix $\delta > 0$ use again the lip. fct. as above and define a function $U_\delta : [0,1] \to \mathbb{R}$ s.t.

$U_\delta = U$ on $[\delta, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1 - \delta]$ $\quad U_\delta(0) = U_\delta(1) = 0 \quad |U_\delta| \leq \frac{1}{2}$

$|U_\delta'| \leq 2$



We then extend U_δ to a 1-periodic $\tilde{U}_\delta$ ~~lip.~~ and set

$u_{n,\delta}(x) = \frac{1}{n} \tilde{U}_\delta(nx)$ Observe $|u_{n,\delta}'| \leq 2$ and

$|u_{n,\delta}'| = 1$ on a set $A_{n,\delta} = \bigcup_{k=0}^{n-1} [\frac{k}{n} + \frac{\delta}{n}, \frac{2k+1}{n} - \frac{\delta}{n}] \cup [\frac{2k+1}{n} + \frac{\delta}{n}, \frac{k+1}{n} - \frac{\delta}{n}]$

which measure is $1-4\delta$. We then have

$$\mathcal{E}(u_{n,\delta}) = \int_{[0,1]\setminus A_{n,\delta}} ||v_{n,\delta}'|^2 - 1| + \int |v_{n,\delta}|^2 \leq 12\delta + \frac{1}{4n^2}$$

This shows $\inf \mathcal{E} \leq 12\delta$ $\delta > 0$ $\inf \mathcal{E} = 0$.

But the existence result page 11 of the notes

Some examples

(i) Consider $f(p) = e^{-p^2}$ and $\inf\{\mathcal{E}(u) \mid u \in X\}$ where $\underset{m}{\shortparallel}$

$X = W_0^{1,1}$

Assume for a moment that $m=0$ then the no function $u \in X$ can satisfy $\int e^{-(u')^2} = 0$ and hence the problem has no solution.

Let us now show that $m=0$. Let $n \in \mathbb{N}$ and define

$$u_n(x) = n(x-\tfrac{1}{2})^2 - \frac{n}{4}$$

then $u_n \in X$ and

$$\mathcal{E}(u_n) = \int_0^1 e^{-(2n(x-\frac{1}{2}))^2} = \frac{1}{2n}\int_{-n}^{n} e^{-y^2} dy \to 0 \quad \Rightarrow \quad n \to \infty$$

(ii) $\mathcal{L}(z,p) = \sqrt{z^2+p^2}$ and

$$\inf_X \mathcal{E}(u) = m \quad \text{where} \quad X = \{u \in W^{1,1}(0,1) \mid u(0)=0,\ u(1)=1\}$$

Let us prove that the pb. has no solution. - We first show that $m \geq 1$ and start by observing $m \geq 1$ since

$$\mathcal{E}(u) \geq \int |u'| dx \geq \int_0^1 u' = u(1) - u(0) = 1.$$

to establish that $m = 1$ we build a sequence min.

$$u_n(x) = \begin{cases} 0 & x \in [0, 1-\frac{1}{n}] \\ 1 + n(x-1) & x \in (1-\frac{1}{n}, 1] \end{cases}$$