CHAPTER III : EXISTENCE THEORY

Throughout this chapter we will fix ~~notation~~ a Lipschitz domain $\Omega \subset \mathbb{R}^n$. The general setting we wish to study is the following:

$$\inf_{u \in \mathcal{X}} \mathcal{E}(u) \quad \text{where} \quad \mathcal{E}(u) = \int \mathcal{L}(x, u, Du)\, dx \quad \text{and}$$

$$\mathcal{X} = \{ u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = g \}$$

where $p \in (1, \infty)$ and $g \in W^{1-\frac{1}{p}, p}(\partial\Omega)$. The continuity assumption on $\mathcal{L}$ are summarised in saying that $\mathcal{L}$ is a CARATHEODORY integrand:

<u>def</u> $\mathcal{L} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is CARATHÉODORY integrand if

- $\mathcal{L}(\cdot, z, p)$ is measurable for all $(z, p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$
- $\mathcal{L}(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.

(III.1) Existence for the Dirichlet energy

We begin our study by considering the model case of Dirichlet energy. We consider the problem

$$\inf \int_\Omega |Du|^2\, dx \qquad \mathcal{X} = \{ u \in W^{1,2}(\Omega) \mid u - g \in W_0^{1,2}(\Omega) \}$$

where $g \in W^{1,2}(\Omega)$

Note that $\mathcal{E}(u) \geq 0$ and $\mathcal{E}(g) < +\infty$ ~~for~~. In particular we have

$$0 \leq \inf_{u \in \mathcal{X}} \mathcal{E}(u) = m < +\infty$$

Thus we may find a sequence $(u_n) \subset \mathcal{X}$ which

$\mathcal{E}(u_n) \to m$. Due to Poincaré inequality we find

$$\| u_n \|_{W^{1,2}(\Omega)} \lesssim \mathcal{E}(u_n) \leq \sup_n \mathcal{E}(u_n) < +\infty$$

Thus u_n is bounded in $W^{1,2}(\Omega)$. Note further that $\mathcal{X}$ is an affine, closed subspace of $W^{1,2}(\Omega)$ which is reflexive

and separable.
Hence by Banach-Alaoglu thm u_n admits a (non-relabeled) subsequence such that $u_n \rightharpoonup u$ for some $u \in \bar{X}$.
Now by Fatou lemma we have

$$m = \liminf \int |Du_n|^2 \geq \int |Du|^2$$

(III.2) <u>An abstract existence result</u>

Let X be a complete metric space. Consider $\mathcal{E} : X \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties.

H1) $\mathcal{E}$ is coercive: for all $\Lambda \in \mathbb{R}$ $\{u \in \bar{X} \mid \mathcal{E}(u) < \Lambda\}$ set is sequentially pre-compact that is whenever $u_n \in X$ is such that $\mathcal{E}(u_n) \leq \Lambda$ $\forall n$ ~~also~~ then $\{u_n\}$ has convergent subsequence.

H2) $\mathcal{E}$ is l.s.c.

<u>Thm</u> Let $\bar{X}$ be a complete metric space. Assume $\mathcal{E}$ is coercive and l.s.c. Then

$$\inf_{u \in \bar{X}} \mathcal{E}(u)$$

admits at least a solution.

<u>proof</u> Assume $\exists\, u_0$ s.t. $\mathcal{E}(u_0) < +\infty$. Otherwise any $u \in \bar{X}$ is a solution. Then consider a minimising sequence u_n. In particular $\exists\, \Lambda > 0$ s.t. $\mathcal{E}(u_n) \leq \Lambda$ $\forall n$ and u_n has a converging subsequence s.t. $u_n \to u$. Due to (H2) $\mathcal{E}(u) \leq \liminf \mathcal{E}(u_n) = m$
Thus u is a minimiser! $\square$

However as discussed last time we can usually not work with the strong topology and hence we need to use a version of the previous thm w.r.t. weak convergence.

<u>thm</u> (weak) Let $\bar{X}$ be a reflexive Banach space or a closed affine subset of a reflexive Banach space. Let $\mathcal{E} : X \to \mathbb{R} \cup \{+\infty\}$ and assume

(H1) $\mathcal{E}$ is weakly coercive: for any Λ $\{u \mid \mathcal{E}(u) \leq \Lambda\}$ is weakly precompact.

(H2) $\mathcal{E}$ is w. l.s.c. $\forall \{u_n\} \subset X$ s.t. $u_n \rightharpoonup u$ $\mathcal{E}(u) \leq \liminf \mathcal{E}(u_n)$

Then $\inf_{u \in \bar{X}} \mathcal{E}(u)$ admits at least a solution.

<u>proof</u> proof is exactly as before using that any strongly closed affine subset of a Banach space is also weakly closed!

III.4 <u>Existence for integrand $\mathcal{L}(x,p)$</u>

We start by considering integrand of the form $\mathcal{L} = \mathcal{L}(x,p)$. In light of the above <u>thm</u> we need to establish weak coercivity and l.s.c. Before doing so we need to check that Caratheodory integrals are Lebesgue-measurable and hence $\mathcal{E}$ is well-defined.

<u>Lemma</u> Let $\mathcal{L} : \Omega \times \mathbb{R}^{n \times m} \to \mathbb{R}$ be Caratheodory. Then for any Borel-measurable $V : \Omega \to \mathbb{R}^{n \times m}$, $x \mapsto \mathcal{L}(x, V(x))$ is Leb-measurable.

<u>proof</u> Suppose first V is a simple function $V = \sum^m v_k \mathbb{1}_{E_k}$ where $E_k \subset \Omega$ are pairwise-disjoint Borel measurable set such that $\Omega = \bigcup^m E_k$ and $v_k \in \mathbb{R}^{n \times m}$ Then

$$\{x \in \Omega \mid \mathcal{L}(x, V(x)) > t\} = \bigcup^m \{x \in E_k \mid \mathcal{L}(x, v_k) > t\}$$

The right-hand side is a union of sets that are Lebesgue-measurable since $\mathcal{L}(\cdot, \sigma_n)$ is Leb.-measurable.
For a general V approximate V by simple functions V_n so that
$$\mathcal{L}(x, V_n(x)) \to \mathcal{L}(x, V(x))$$
pointwise in Ω. Thus $\mathcal{L}(x, V(x))$ is a pointwise limit of Lebesgue measurable functions and hence Leb.-meas. $\square$

In general it is possible that $x \mapsto \mathcal{L}(x, u, Du)$ is meas. but $\mathcal{E}(u)$ is not-well defined. This can be avoided if $\mathcal{L} \geq 0$ or
$$\mathcal{L}(x, z, p) \lesssim (1 + |z|^{\alpha} + |p|^{\alpha})$$

~~This growth assumption where was~~

The most common used coercivity assumption is the existence of $\lambda > 0$ s.t.
$$\mathcal{L}(x, p) \geq \lambda |p|^{\alpha} \qquad (C)$$

Often this condition is stated as
$$\mathcal{L}(x, p) \geq \lambda |p|^{\alpha} - c$$
for some $c > 0$. However by setting $\tilde{\mathcal{L}}(x, p) = \mathcal{L}(x, p) + c$ we recover (C) without changing minimizers.
Further (C) specify the space $W^{1,\alpha}(\Omega)$ in which we look for solution!

<u>prop</u> If $\mathcal{L} : \Omega \times \mathbb{R}^{n \times m} \to [0, +\infty)$ is Carathéodory and satisfies (C) for some $\lambda > 0$ and $\alpha \in (1, +\infty)$. Then $\mathcal{E}$ is weakly coercive on $W^{1,\alpha}_g := \{ u \in W^{1,\alpha} \mid u|_{\partial\Omega} = g \}$

<u>proof</u> Let $\{u_n\} \subset W^{1,\alpha}$ and assume $\mathcal{E}(u_n) < +\infty$.
Due to (4.2) we have
$$\lambda \sup_n \int |Du_n|^{\alpha} \leq \sup_n \mathcal{E}(u_n) < +\infty$$
~~Fix~~ By applying poincaré inequality to $u - g$ we get
$$\| u_n - g \|_{L^{\alpha}} \leq \| u_n - g \|_{L^{\alpha}} + \| g \|_{L^{\alpha}}$$

$$\leq c\|Du_n - Dg\|_{L^\alpha} + \|g\|_{L^\alpha}$$
$$\lesssim \|g\|_{W^{1,\alpha}} + \underbrace{\|Du_n\|_{L^\alpha}}_{\text{bounded by coercivity}}$$

As $\alpha > 1$ this means that u_n is bounded in $W^{1,\alpha}_g$ a closed affine subspace of $\underbrace{W^{1,\alpha}(\Omega)}_{\text{reflexive and sep space}}$. This implies that $\mathcal{E}(u)$ is weakly coercive

□

We next turn towards establishing l.s.c. . A first result in this sense is due to Tonelli ($n=1$) and Serrin ($n>1$). It shows that convexity implies l.s.c.

<u>Thm</u> Let $\mathcal{L} : \Omega \times \mathbb{R}^{n\times N} \to [0, +\infty)$ be carathéodory and such that $\mathcal{L}(x, \cdot)$ is convex for almost every $x \in \Omega$. Then $\mathcal{E}$ is l.s.c. on $W^{1,\alpha}$ for any $\alpha \in (1, \infty)$

<u>Mazur Lemma</u> (Mazur) Let $(X, \|\cdot\|)$ be a normed space and let $x_n \rightharpoonup x$ in X. Then $\exists \{y_n\} \subset co\{x_n\}$ s.t. $y_n \to x$ in X.

More precisely for every n $\exists\, m_n$ and $\alpha_n^i > 0$ $\sum^{m_n} \alpha_n^i = 1$ s.t.
$$y_n = \sum^{m_n} \alpha_n^i x_i \quad \text{and}$$
$$\|y_n - x\| \to 0 \quad \text{as} \quad n \to +\infty$$

<u>proof</u> (thm)

<u>step 1</u> We first prove that $\mathcal{E}$ is strongly sequentially l.s.c.

Suppose $u_n \to u$ in $W^{1,\alpha}$. After passing to a subsequence we may assume $Du_n \to Du$ a.e. Note that $\mathcal{L}(x, Du_n) \geq 0$ so that by Fatou's lemma.
$$\mathcal{E}(u) = \int \mathcal{L}(x, Du) \leq \liminf \int \mathcal{L}(x, Du_n)$$
$$= \liminf \mathcal{E}(u_n)$$

As this holds for all subsequences we deduce l.s.c.

Step 2 Suppose $\{u_n\} \subset W^{1,\alpha}$ and $u \in W^{1,\alpha}$ are s.t.

$u_n \rightharpoonup u$ in $W^{1,\alpha}$. The $\exists$ a subsequence such that

$$\mathcal{E}[u_n] \to \liminf \mathcal{E}[u_n] = m.$$

By Mazur's Lemma $\exists$

$$v_n = \sum_{j=n}^{N(n)} \alpha_j^n u_j \qquad \sum^{N(n)} \alpha_j^n = 1$$

such that $v_n \to u \in W^{1,\alpha}$. On the one hand $\mathcal{L}(x,\cdot)$ is convex a.e.

$$\mathcal{E}(v_n) \leq \int \mathcal{L}(x, \sum_j^{N(n)} \alpha_j D u_j)\, dx$$

$$\leq \sum \alpha_j^n \int \mathcal{L}(x, Du_j) \to m$$

On the other hand by Step 1 we have l.s.c. strongly.! $\square$

Combining the previous theorems we get the following existence results

thm (Existence) Let $\alpha \in (1,\infty)$ and $\mathcal{L}: \Omega \times \mathbb{R}^{n \times m} \to [0,+\infty)$ be Carathéodory s.t.

(i) $\mathcal{L}$ is coercive : $\mathcal{L}(x,p) \geq \lambda |p|^\alpha$ a.e. x and $\forall p$

(ii) $\mathcal{L}(x,\cdot)$ is convex a.e. $x \in \Omega$

Then $\mathcal{E}$ has a minimizer on $W_g^{1,p}(\Omega)$.

We now establish that convexity is the sharp assumption to obtain weak l.s.c. in the scalar case ~~or~~ one dimensional case when $\mathcal{L}$ is independent of x.

prop Let $\mathcal{E}: W^{1,\alpha}(\Omega) \to \mathbb{R}$ be such that $\mathcal{L} = \mathcal{L}(p): \mathbb{R}^{m \times n} \to \mathbb{R}$ is continuous. If $\mathcal{E}$ is weak l.s.c. on $W^{1,\alpha}$ and $(m=1)$ or $(n=1)$ then $\mathcal{L}$ is convex.!

proof We focus on the case $m=1$ $n\geq 1$ (the case $n=1$ can be dealt similarly) Fix $a,b\in\mathbb{R}^n$ $a\neq b$ and $\theta\in(0,1)$

$v = \theta a + (1-\theta)b$ Introduce

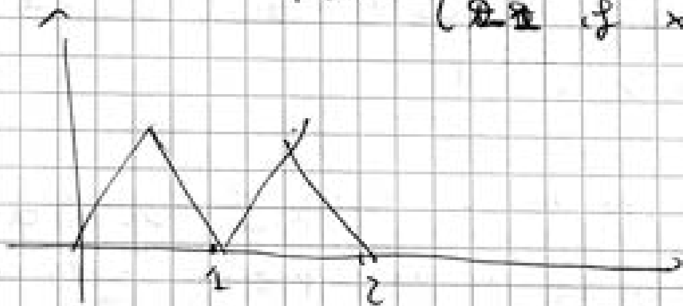
$$\cancel{u_j(x) = \langle v, x\rangle + \frac{1}{j}\phi_0(\langle jx, n\rangle - \lfloor jx\, n\rfloor}\ )$$

where

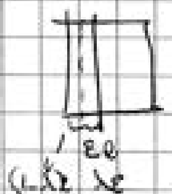
$$\cancel{\Phi_0(x) = \begin{cases} -j(1-\theta)x & x\in(0,\theta) \\ \theta x - \theta & x\in[\theta,1) \end{cases}}$$

let us define a sequence in the following way: define $\phi:\mathbb{R}\to\mathbb{R}$ to be 1 periodic function with slope ϕ' that satisfies

$$\phi'(x) = \begin{cases} \cancel{x} & \text{if } x\in\mathbb{Z}+[0,1-\lambda] \\ \cancel{x} & \text{if } x\in\mathbb{Z}+[1-\lambda,1], \end{cases}$$



set $u_\varepsilon(x) = \varepsilon\phi\left(\frac{\langle x, b-a\rangle}{\varepsilon}\right) + \langle v\cdot x\rangle$ Then u_ε converges uniformly to $\langle v, x\rangle$ and hence in $L^q(\Omega)$ Furthermore the gradient is given by



$$\nabla u_\varepsilon(x) = v + \phi'\left(\frac{\langle x, b-a\rangle}{\varepsilon}\right)(b-a) = \begin{cases} b & x\in\Omega\setminus A_\varepsilon \\ a & x\in A_\varepsilon \end{cases}$$

where A_ε is the union of stripes of thi thickness $\lambda\varepsilon$.

Then ∇u_ε is uniformly bounded in $L^\infty(\Omega)$ and hence in every $L^p(\Omega)$ and we deduce $u_\varepsilon \rightharpoonup \langle v, x\rangle$ in $W^{1,\infty}$

By l.s.c.

$$|\Omega| \mathcal{L}(v) = \mathcal{E}(\langle x; v\rangle) \leq \liminf \mathcal{E}(u_\varepsilon) =$$
$$= |\Omega|\left(\theta\mathcal{L}(a) + (1-\theta)\mathcal{L}(b)\right)$$

$\square$

We finally end this section with the following uniqueness result.

prop. Let $\alpha\in(1,\infty)$. Consider $\mathcal{E}: W^{1,\alpha}(\Omega)\to\mathbb{R}$ with $\mathcal{L}:\Omega\times\mathbb{R}^{m\times n}\to\mathbb{R}_+$. If $\mathcal{L}(x,\cdot)$ is strictly cvx a.e. $x\in\Omega$. Then a minimizer $u\in W^{1,\alpha}_g(\Omega)$ is unique if it exists.

(III.5) <u>Existence for integrands with u-dependence.</u>

$$\mathcal{E}(u)=\int_\Omega \mathcal{L}(x,u,Du)\,dx$$

We will make the coercivity assumption

$$\lambda|p|^\alpha + C_1|z|^q + C_2 \leq \mathcal{L}(x,z,p)$$

for some $\lambda>0$, $C_1,C_2\in\mathbb{R}$ and $\alpha>q\geq 1$ and almost every $x\in\Omega$, all $(z,p)\in\mathbb{R}^m\times\mathbb{R}^{m\times n}$!

Thus one can easily prove as before that $\mathcal{E}$ is weakly coercive. What about l.s.c.? If we try to prove l.s.c. as we did before we encounter the expression

$$\int_\Omega \mathcal{L}\Big(x,\sum^{N(n)}\theta_n^j u_j,\sum^{N(n)}\theta_n^j Du_j\Big)\,dx$$

whereas before we were able to pull the sum outside of using convexity, without assuming convexity of $\mathcal{L}(x,\cdot,\cdot)$ we cannot do so anymore. Nevertheless, sequential weak lower semicontinuity does hold!

<u>Thm</u> Suppose $\mathcal{L}:\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to[0,+\infty)$ is Carathéodory. Assume $\mathcal{L}(x,z,\cdot)$ is convex for every $(x,z)\in\Omega\times\mathbb{R}^m$. Then for every $\alpha\in(1,\infty)$ $\mathcal{E}$ is l.s.c. on $W^{1,\alpha}$

<u>proof</u> We will prove the theorem under the additional assumption that $\mathcal{L} \in C^1$ convex in (z,p) and that $\exists C > 0$ such that for all $(x,z,p) \in \bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$ it holds

$$\mathcal{L}(x,z,p) \geq \lambda |p|^\alpha$$

$$|\partial_z \mathcal{L}(x,z,p)| + |\partial_p \mathcal{L}(x,z,p)| \leq C\,(1 + |z|^{\alpha-1} + |p|^{\alpha-1})$$

The proof without this assumption is considerably more technical.

Let $\{u_n\} \subset W^{1,p}_g(\Omega)$ and $u \in W^{1,p}_g(\Omega)$ be such that $u_n \to u$ weakly in $W^{1,p}$. Using convexity and C^1-regularity of $\mathcal{L}$ we find

$$f(x,u_n,Du_n) \geq f(x,u,Du) + \mathcal{L}_z(x,u,Du)(u_n - u_n) \qquad (*)$$
$$+ \langle \mathcal{L}_p(x,u,Du),(Du_n - Du)\rangle$$

We want now to integrate this expression. In order to do so we need to check that

$$(**) \qquad \mathcal{L}_z(x,p,u)(u_n - u) + \langle \mathcal{L}_p(x,u,Du), Du_n - Du\rangle \in L^1(\Omega)!$$

Using Hölder's inequality

$$\int_\Omega |\mathcal{L}_z(x,u,Du)|\,|u_n - u| \leq \left(\int \left(1 + |u|^{\alpha-1} + |Du|^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \|u_n - u\|_{L^\alpha}$$
$$\leq (1 + \|u\|^\alpha_{L^\alpha} + \|u_n\|^\alpha_{L^\alpha}) < +\infty \quad ?$$

the other term is estimated similarly. In particular integrating $(*)$ over Ω

$E(u_n) \geq E(u) + \int_\Omega (\quad\text{———}\quad)$ However as $u_n \to u$ in $W^{1,\alpha}$

the last two terms converges to 0 and we deduce

$$\liminf E(u_n) \geq E(u)$$

thm Supposed the assumptions of the previous thm holds and that $\mathcal{L}$ is α-coercive. $\alpha \in (1,\infty)$ Then there exists at least one minimizer of $\inf_{u \in X} \mathcal{E}(u)$ $X = W_g^{1,\alpha}(\Omega)$

Again about sharpness of convexity

thm Let $\mathcal{E}(u) = \int \mathcal{L}(x,u,Du)$ with u either scalar ($m=1$) or one dimensional (u is defined on $[a,b]$) and assume that

(i) $\mathcal{L}$ cont. w.r.t. all variables

(ii) $\mathcal{E}$ w.l.s.c. on $W_g^{1,p}(\Omega)$ for some α and g

Then $\mathcal{L}(x,z,\cdot)$ is cvx for every x and z

proof In the scalar case Fix $\bar{x} \in \Omega$ and $\bar{z}$ and consider the convex combination $\sigma = \lambda a + (1-\lambda) b$

we want to show

$$\mathcal{L}(\bar{x},\bar{u},\sigma) \leq \lambda \mathcal{L}(\bar{x},\bar{u},a) + (1-\lambda)\mathcal{L}(\bar{x},\bar{u},b)$$

Fix $\eta > 0$ We claim that $\exists$ Ω' neighborhood of $\bar{x}$ and $\delta > 0$ s.t.

$$|\mathcal{L}(x,z,p) - \mathcal{L}(\bar{x},\bar{z},p)| \leq \eta \quad \forall x \in \Omega', \ |z-\bar{z}| \leq \delta \ \text{and} \ \varphi \in B(\sigma,R)$$

with R big enough for a and b to belong to the ball.

Choose $u(x) = \bar{z} + \sigma x$ for all $x \in \Omega'$ and

$$\varphi_\varepsilon(x) = \varepsilon \varphi\left(\frac{x(b-a)}{\varepsilon}\right)$$

Let σ_ε a cut off function s.t. $\sigma_\varepsilon \equiv 1$ on Ω' and $\sigma_\varepsilon(x) = 0$ if $d(x,\Omega') \geq \sqrt{\varepsilon}$. The app. sequence is given by

$$u_\varepsilon(x) = u + \sigma_\varepsilon \varphi_\varepsilon \quad \text{and therefore}$$

$$\mathcal{E}(u) = \int_{\Omega \setminus \Omega'} \mathcal{L}(x,u,\nabla u)\,dx + \int_{\Omega'} \mathcal{L}(x,u,\bar{\sigma})\,dx$$

$$\geq \int_{\Omega\setminus\Omega'} (\ - \ ) + \int |\Omega'|\, \mathcal{L}(\bar{x},\bar{z},\sigma) - \eta|\Omega'|$$

On the other hand

$$\mathcal{E}(w_\varepsilon) \geq \int_{\Omega\setminus(\Omega'\cup B_\varepsilon)} \mathcal{L}(x,u,\nabla u)dx + \int_{B_\varepsilon} \mathcal{L}(x,w,\sigma)dx$$

$$+ \int_{A_\varepsilon} \mathcal{L}(x,w,a)dx + \int_{\Omega'\setminus A_\varepsilon} \mathcal{L}(x,w,b)dx$$

$$\leq \int_{\Omega\setminus(\Omega'\cup B_\varepsilon)} f(x,u,\nabla u)dx + \int_{B_\varepsilon} \mathcal{L}(x,u,\sigma)dx$$

$$+ \mathcal{L}(\bar{x},\bar{u},a)\,|A_\varepsilon| + \mathcal{L}(\bar{x},\bar{u},b)\,|\Omega'\setminus A_\varepsilon| + \eta\,|\Omega'|$$

taking the limit as $\varepsilon \to 0^+$ we obtain the estimate

$$\mathcal{L}(\bar{x},\bar{u},\xi)\,|\Omega| - \eta|\Omega'| \leq \lambda\mathcal{L}(\bar{x},\bar{u},a)|\Omega'|$$
$$+ (1-\lambda)\mathcal{L}(\bar{x},\bar{u},b)\,|\Omega'|$$
$$+ \eta\,|\Omega'|$$

<u>EXAMPLE</u> (LINEARISED ELASTICITY) The thm above is a powerful tool to handle a large class of problems. It is useful in order to establish a existence theory also for problemi which one coercive are we asked but not of the form $\mathcal{L}(x,u,Du)$.
Suppose we are given a body in a reference configuration Ω. We now deform Ω using elastic deformation $y: \Omega \to y(\Omega)$. Since it is elastic y should be a diff. bijection that is orientation preserving i.e. $\det Dy > 0$ for $x \in \Omega$. It is natural to describe the energy in terms of the deformation $u(x) = y(x) - x$. We expect the energy associated to the configuration $y(\Omega)$ to be preserved under rigid motions of Ω. A wise energy is the Green St. Venant stress tensor $G(u) = \frac{1}{2}(Du + Du^T + Du^T Du)$ notice that $\mathcal{L}(u) = Du + Du^T$ is the symmetric part of the gradient.

If we assume Du is small $G(u) \sim \frac{1}{2}\mathcal{L}(u)$

Thus in linearised elasticity a good basic model is given by

$$u \longmapsto \frac{1}{2}\int \langle \mathcal{E}(u), C(x)\mathcal{E}(u)\rangle \, dx$$

Here $C(x)$ is a positive-definite fourth order tensor known as the elasticity tensor. If the medium is homogeneous and isotropic $C(x) \equiv C$ and it is possible to show that the energy above reduces to

$$\mathcal{E}(u) := \quad u \to \int \mu |\mathcal{E}(u)|^2 + \frac{1}{2}\left(k - \frac{2}{3}\mu\right)|\mathrm{tr}\, \mathcal{E}(u)|^2 \, dx$$

Here k is the bulk constant while μ is called the shear constant $\rightsquigarrow$ they describe properties of the medium.

Consider hence the min pb.

$$\inf \left\{ \mathcal{E}(u) \,\Big|\, \frac{1}{2}\int \mu |\mathcal{E}(u)|^2 + \frac{1}{4}\left(k - \frac{2}{3}\mu\right)|\mathrm{tr}\, \mathcal{E}(u)|^2 - \langle b, u\rangle \, dx \right.$$
$$\left. \Big| u \in X \right\}$$

where $X = \{ W^{1,2}(\Omega; \mathbb{R}^2) \mid u\big|_{\partial\Omega} = g \}$

Here $b \in C^c$ describes an external force we apply to the body.

Note that $\mathcal{E}$ has a quadratic growth and is convex in $\mathcal{E}(u)$.

In order to get existence we have to show the $\mathcal{E}$ is coercive.

Assume $g = 0$ and $k - \frac{2}{3}\mu \geq 0$. Note that for any ϕ

$$2\langle \mathcal{E}(\phi) : \mathcal{E}(\phi)\rangle - \langle D\phi, D\phi\rangle = \mathrm{div}(D\phi\,\phi - \mathrm{div}\,\phi\,\phi) + \mathrm{div}(\phi)^2$$

Integrating over Ω and applying the divergence thm we find.

$$2\|\mathcal{E}(\phi)\|^2_{L^2} + 2\|D\phi\|^2_{L^2} = \int \mathrm{div}(D\phi\,\phi - \mathrm{div}(\phi)\phi) + \mathrm{div}(\phi)^2 \, dx$$
$$\geq \int \mathrm{div}(\phi)^2 dx = 0$$

$\hookrightarrow$ Korn's inequality

Then using Hölder inequality we deduce

$$\mathcal{E}(u) \geq \mu \|\mathcal{E}(u)\|^2_{L^2} - \|b\|_{L^2}\|u\|_{L^2}$$

$$\geq \mu \|Du\|^2_{L^2} - C(\delta)\|b\|^2 - \delta\|u\|^2_{L^2}$$

$$\geq \frac{\mu}{2}\|Du\|^2_{L^2} - C\|b\|^2_{L^2} \text{ by choosing } \delta \text{ small}$$

$\Rightarrow \mathcal{E}$ is coercive !

(III.6) Integral side constraint

Notice that so far we have only considered Dirichlet boundary conditions. However in one of the previous lecture we have introduce other side conditions: in particular $\int h(x,u) = 0$ for some function h. We want now adapt the existence thm to this case beginning with ~~a~~ an abstract existence thm!

Thm Let X a Banach space or a closed affine subset of a Banach space. Suppose $\mathcal{E}, G : X \to \mathbb{R} \cup \{+\infty\}$. Assume

H1) $\mathcal{E}$ is weakly coercive

H2) $\mathcal{E}$ is l.s.c.

H3) G is weakly cont. $u_n \rightharpoonup u$ then $G(u_n) \to G(u)$

Assume $\exists u_0$ s.t. $G(u_0) = 0$ then the pb.

$$\inf \{\mathcal{E}(u) \mid u \in X \text{ and } G(u) = 0\}$$

admits at least a solution.

The following lemma gives the ~~right~~ required weak continuity for an integral side constraint.

Lemma Let $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ Carathéodory. Suppose for some $\alpha \in (1, \infty)$ and $C > 0$ we have

$$|h(x,z)| \leq C(1 + |z|^q) \quad \text{a.e. } x \in \Omega$$

and every $y \in \mathbb{R}^m$ where $q \in (1, \frac{n\alpha}{n-\alpha})$ if $\alpha < n$ and $q <$ +∞

if $\alpha \geq n$. Then $G(u)$ is weakly cont.

Proof We deal with the case $\alpha < n$ ($\alpha \geq n$ is similar).
Suppose $u_k \rightharpoonup u$ in $W^{1,\alpha}$. Up to passing to a subsequence (not relabeled) we may assume $u_k \to u$ in L^q and almost everywhere in Ω. For almost every $x \in \Omega$ we have
Then by Fatou' Lemma we get what we need!

$\square$

Rmk when $G(u) = \int u = 0$ we have Neumann condition

Combining the thm above we get the desired existence thm.

(III.7) EULER-LAGRANGE ~~CONDITION~~ EQUATIONS

We have to consider several growth condition on L and its derivatives

Condition A (Growth on L) L is caratheodory and satisfying for almost every $x \in \Omega$ and for every (z,p)

$$|L(x,z,p)| \leq \alpha(x) + \beta\left(|z|^\alpha + |p|^\alpha\right)$$

$\alpha \in L^1(\Omega)$ and $\beta \geq 0$

Condition (Growth condition I) All the partial derivatives of L in p and z are Caratheodory. Moreover they satisfy for a.e. $x \in \Omega$ and every (z,p)

$$|L_z(x,z,p)|, |L_p(x,z,p)| \leq \alpha_1(x) + \beta\left(|z|^\alpha + |p|^\alpha\right)$$

$\alpha_1 \in L^1(\Omega)$, $\beta \geq 0$

Condition (Growth condition II) All partial derivative Caratheodory Moreover for every $R > 0$ $\exists$ $\alpha_3 \in L^1(\Omega)$ and $\alpha_2 \in L^{\alpha/\alpha-1}(\Omega)$ and $\beta = \beta(R) > 0$ s.t. for a.e. $x \in \Omega$ and every $(z,p) \in B_R \times \mathbb{R}^{d \times m}$

$$|\mathcal{L}_z(x,z,p)| \le \alpha_1(x) + \beta|p|^\alpha$$

$$|\mathcal{L}_p(x,z,p)| \le \alpha_2(x) + \beta|p|^{\alpha-1}$$

where $B_R := \{ t \in \mathbb{R}^m \mid |t| \le R \}$

<u>Condition (Growth condition III)</u> Partial der. Caratheodory.
For a.e. $x \in \Omega$ and every (z,p)

$$|\mathcal{L}_z(x,z,p)| \le \alpha_1(x) + \beta\left(|z|^{\alpha-1} + |p|^{\alpha-1}\right)$$

$$|\mathcal{L}_p(x,z,p)| \le \alpha_2(x) + \beta\left(|z|^{\alpha-1} + |p|^{\alpha-1}\right)$$

$\alpha_1, \alpha_2 \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$ and $\beta \ge 0$.

We are now ready for the main thm of this chapter. Notice that it is only based on sensible applications of Hölder inequality and Sobolev embedding thm.

<u>thm</u> (Weak ELE) Let $\mathcal{L}$ be as in <u>Condition A</u> and for $\varphi : \Omega \to \mathbb{R}^N$ let.

$$\delta\mathcal{E}(u,\varphi)\ L(u,\varphi) = \int \left(\langle \mathcal{L}_p(x,u,Du); D\varphi \rangle + \langle \mathcal{L}_z(x,u,Du); \varphi \rangle \right) dx$$

Assume that $\bar{u} - g \in W_0^{1,\alpha}(\Omega)$ is a minimizer of

$$\inf \left\{ \mathcal{E}(u) \mid u - g \in W_0^{1,\alpha}(\Omega) \right\}$$

(I) If growth condition (I) holds then

$$\cancel{L(u,\varphi)}\ \delta\mathcal{E}(u,\varphi) = 0 \qquad \forall \varphi \in C_0^\infty(\Omega) \quad (*)$$

(II) If growth condition (II) holds and $\bar{u} \in L^\infty(\Omega)$ then

$$\delta\mathcal{E}(u,\varphi) = 0 \quad \forall \varphi \in W_0^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$$

(III) (III) holds then

$$\delta\mathcal{E}(u,\varphi) = 0 \quad \forall \varphi \in W_0^{1,\alpha}(\Omega)$$

Conversely, if $\bar u$ satisfyes ⊛ and $(z,p) \mapsto \mathcal{L}(x,z,p)$ is convex for almost every $x \in \Omega$ then $\bar u$ is a minimizer of the pb.

proof

step 1 Notice that because of the growth condition on $\mathcal{L}$ then for every $\varphi \in W^{1,\alpha}$ $\mathcal{E}(\bar u + \varepsilon\varphi)$ is well defined. $\varepsilon \in \mathbb{R}$

Since $\bar u$ is a minimizer then

$$\mathcal{E}(\bar u + \varepsilon\varphi) \geq \mathcal{E}(\bar u) \qquad \forall \varphi \in C_c^\infty \quad (I)$$

$$\forall \varphi \in W_0^{1,\alpha} \cap L^\infty \quad (II)$$

$$\forall \varphi \in W_0^{1,\alpha} \quad (III)$$

We thus have if the limit exists that

$$\delta\mathcal{E}(u,\varphi) = \lim_{\varepsilon \to 0} \frac{\mathcal{E}(\bar u + \varepsilon\varphi) - \mathcal{E}(\bar u)}{\varepsilon} = 0$$

Let us show the limit exists

$$g(x,\varepsilon) = \int_0^1 \langle D_z\mathcal{L}(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); \varphi \rangle + \langle \mathcal{L}_p(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); D\varphi \rangle \, dt$$

and we find.

$$\frac{\mathcal{E}(\bar u + \varepsilon\varphi) - \mathcal{E}(\bar u)}{\varepsilon} = \frac{1}{\varepsilon}\int_\Omega dx \int_0^1 \frac{d}{dt}[\mathcal{L}(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi)] \, dt =$$

$$= \frac{1}{\varepsilon}\int_\Omega g(x,\varepsilon)\, dx$$

If we can show $\exists \eta \in L^1(\Omega)$ s.t. for every ε small enough

$$|g(x,\varepsilon)| \leq \eta(x) \quad \text{a.e. } x \qquad (*)$$

we get the limit by applying Leb. dominated cv thm.

let us show $(*)$!

Growth condition (I) Since $\varphi \in C_c^\infty(\Omega)$ we find that.

$$|\langle L_u(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); \varphi\rangle|$$
$$\leq [\alpha_1(x) + \beta(|\bar u + \varepsilon t\varphi|^\alpha + |D\bar u + \varepsilon t D\varphi|^\alpha)]\,|\varphi|$$

$$|\langle L_p(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); D\varphi\rangle|$$
$$\leq [\alpha_1(x) + \beta(|D\bar u + \varepsilon t D\varphi|^\alpha + |\bar u + \varepsilon t\varphi|^\alpha)]\,|D\varphi|$$

Summing up the two inequalities and taking the sup in $(t, \varepsilon) \in [0,1] \times [-1,1]$ we get ⊛.

Growth condition II Since $\bar u, \varphi \in L^\infty$ we can find $R > 0$ so that for every $(t, \varepsilon) \in [0,1] \times [-1,1]$

$$|\bar u + t\varepsilon\varphi| \leq R \quad \text{a.e. } \Omega$$

we therefore find

$$|\langle L_u(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); \varphi\rangle| \leq [\alpha_1(x) + \beta|D\bar u + t\varepsilon D\varphi|^\alpha]\,|\varphi|$$

$$|\langle L_p(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); D\varphi| \leq [\alpha_2(x) + \beta|D\bar u + t\varepsilon D\varphi|^{\alpha-1}]\,|D\varphi|$$

Noting that, since $\bar u, \varphi \in W^{1,p}(\Omega) \cap L^\infty$ we have by Hölder.

$$\alpha_1|\varphi|,\ |D\bar u + t\varepsilon D\varphi|^\alpha|\varphi| \in L^1$$
$$\alpha_2|D\varphi|,\ |D\bar u + t\varepsilon D\varphi|^{\alpha-1}|D\varphi| \in L^1$$

As before we get the desired result.

Growth condition III We find

$$|\langle L_u(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); \varphi\rangle|$$
$$\leq [\alpha_1(x) + \beta(|\bar u + t\varepsilon\varphi|^{\alpha-1} + |D\bar u + t\varepsilon D\varphi|^{\alpha-1})]\,|\varphi|$$

$$|\langle L_p(x, \bar u + t\varepsilon\varphi, D\bar u + t\varepsilon D\varphi); D\varphi\rangle|$$
$$\leq [\alpha_2(x) + \ldots\ldots \qquad]\,|D\varphi|$$

Again $\quad \alpha_1|\varphi|,\ |\bar u + t\varepsilon\varphi|^{\alpha-1}|\varphi|,\ |D\bar u + t\varepsilon D\varphi|^{\alpha-1}|D\varphi| \in L^1(\Omega)$

Step 2 It remains to prove, provided $(z,p) \mapsto \mathcal{L}(x,z,p)$ is convex, then any solution $\bar{u}$ of EL is a minimizer of (P).

$$\mathcal{L}(x,u,Du) \geq \mathcal{L}(x,\bar{u},D\bar{u}) + \langle \mathcal{L}_p(x,\bar{u},D\bar{u}), D(u-\bar{u})\rangle$$
$$+ \langle \mathcal{L}_z(x,\bar{u},D\bar{u}), u-\bar{u}\rangle$$

therefore for any $\bar{u}-g \in W_0^{1,\alpha}(\Omega)$ we have after integration and appealing to EL $(u-\bar{u} \in W^{1,\alpha})$

$$\int_\Omega \mathcal{L}(x,u,Du)dx \geq \int \mathcal{L}(x,\bar{u},D\bar{u})\,dx$$

(III.8) YOUNG MEASURES

Remind the following l.s.c. thm.

Thm Let $\mathcal{E}(u) = \int \mathcal{L}(x,u,Du)\,dx$ $\mathcal{L}: \Omega \times \mathbb{R}^m \times \mathbb{R}^{d\times m} \to [0,+\infty)$ Carathéodory. Assume $\mathcal{L}(x,z,\cdot)$ convex for almost every $x \in \Omega$ and every $z \in \mathbb{R}^m$. Then $\mathcal{E}$ is lsc. on $W^{1,\alpha}(\Omega)$ $\alpha \in (1,\infty)$.

We want to prove this result by a powerful tool known as Young measures.

Let u_n be a sequence of function converging in measure (pointwise a.e.) to some u. Then for all g continuous we have $g \circ u_n \to g \circ u$.

But if u_n converges weakly to one might wonder what happens to the limit of the composition. If g is affine is true but if g is general this is not true anymore.

ex Let $g(s) = s^2$ and let u_n be a sequence of oscillating functions taking values $\{\pm 1\}$ that converges to zero. Then $g \circ u_n = 1 \to 1$ but $g \circ u$ is 0 and hence convergence fails

Setting let Ω be an open set in $\mathbb{R}^d$, K a compact metric space and let $u_n : \Omega \to K$ be a sequence of functions.
let $\mathcal{P}(K)$ be the space of probability measures on K

def A function $\mu : \Omega \to \mathcal{P}(K)$, $x \mapsto \mu_x$ is weak-$*$ Borel if it is Borel with respect to the weak-$*$ topology on $\mathcal{P}(K)$. In other words the mapping

$$x \mapsto \int_K g(y)\, d\mu_x(y)$$

is Borel for all $g \in \mathcal{C}(K)$

The following results, known as the fundamental theorem for Young measures, shows that each sequence (actually a subsequence) of maps as above generates a weak-$*$ Borel function with specific properties.

Thm let $u_n : \Omega \to K$ be a sequence of maps. Then there are a subsequence u_{n_k} and a w-$*$ Borel map $\mu : \Omega \to \mathcal{P}(K)$ s.t.

(i) For every $g : \Omega \times K \to \mathbb{R}$ s.t. $g(x, \cdot)$ is cont. at almost every $x \in \Omega$ and $\int_\Omega \sup_y |g(x,y)|\, dx < +\infty$ we have

$$\int_\Omega g(x, u_{n_k}(x))\, dx \to \int_\Omega \left(\int_K g(x,y)\, d\mu_x(y) \right) dx$$

(ii) For every cont. map $g : K \to \mathbb{R}$

$$g(u_{n_k}(x)) \xrightarrow{*} \int g(y)\, d\mu_x(y) \quad \text{in } L^\infty$$

(iii) For all $g : \Omega \times K \to \mathbb{R}$ Borel and bounded s.t. $g(x, \cdot)$ is cont. for a.e. $x \in \Omega$ then it turns out that.

$$g(x, u_{n_k}(x)) \xrightarrow{*} \int_K g(x,y)\, d\mu_x(y) \quad \text{in } L^\infty$$

(iv) if $K \subset \mathbb{R}^m$ then $u_{n_k} \xrightarrow{*} u_\infty$ in L^∞ where

$$u_\infty(x) = \int_K y\, d\mu_x(y)$$

v) The measure μ_x equals $\delta_{u_\infty(x)}$ for a.e. $x \in \Omega$ iff u_n converges in measure to u_∞.

The map $x \mapsto \mu_x$ is called the <u>Young measure</u> generated by the family of functions u_n

<u>ex</u> $K = [-1, 1]$ $\Omega = \mathbb{R}$ $u_n(x) = f(nx)$ where f is the 1 periodic function

$$f(x) = \begin{cases} y_1 & 0 \le x < \lambda \\ y_2 & \lambda \le x \le 1 \end{cases}$$

in this case $\mu_x = \lambda \delta_{y_1} + (1-\lambda)\delta_{y_2}$ is the map given by the thm above. Note that

$$g(u_n(x)) \to \lambda g(y_1) + (1-\lambda) g(y_2) \quad \text{for a.e. } x \in \Omega$$

<u>Thm</u> if E separable Banach space then the dual of $L^1(\Omega; E)$ is $L^\infty_w(\Omega; E^*) \to$ Borel function w.r.t weak-* topology on E^*

<u>proof</u> For each $n \in \mathbb{N}$ consider the map defined by setting

$$\mu^n : \Omega \ni x \mapsto \delta_{u_n(x)} \in \mathcal{P}(K)$$

Then $\{\mu^n\}$ is a sequence of maps from Ω to the probability meas. on K which is a subset of $\mathcal{M}(K)$ the signed measure. A well known duality thm asserts that.

$$\mathcal{M}(K) = \left(C(K)\right)^*$$

which means that μ^n can also be considered as a sequence of elements that belongs to $L^\infty_w(\Omega, (C(K))^*)$. This space is the dual of $L^1(\Omega, C(K))$

and since $\|\mu^n\|_\infty = 1$ we can apply Banach-Alaoglu to find a measure μ and a subsequence s.t.

$$\mu^{n_k} \rightharpoonup \mu \quad \text{in} \quad L^\infty_w(\Omega, \mathcal{M}(K)) \quad (*)$$

Now let $G \in L^1(\Omega, C(K))$. we can see G as a function $g : \Omega \times K \to \mathbb{R}$ s.t. $g(x, \cdot)$ is continuous for a.e. $x \in \Omega$ and

$$\int \sup_y |g(x,y)| \, dy < +\infty$$

the convergence $(*)$ can also be written as.

$$\int_\Omega \langle \mu^{n_n}, G \rangle \to \int_\Omega \langle \mu, G \rangle \qquad \forall G \in L^1(\Omega; C(K))$$

which is equivalent

$$\int_\Omega \int_K g(x, u_{n_k}(x)) \, dx \to \int_\Omega \left(\int_K g(x,y) \, d\mu_x(y) \right) dx$$

where g is the function associated to G. this prove (i). to verify $\mu_x \in \mathcal{P}(K)$ for all x plug $g(x,y) = \alpha(x) \in L^1(\Omega)$ into the above eq. It turns out

$$\int_\Omega \alpha(x) \, dx = \int_\Omega \alpha(x) \mu_x(K) \, dx$$

which implies $\mu_x(K) = 1$ a.e. $x \in \Omega$ since α is arbitrary. to prove that μ_x is not a signed measure we simply not that

$$\|\mu^{n_k}\|_\infty = 1 \to \|\mu\|_\infty \leq 1$$

l.sc. of the norm

is non-negative otherwise the total variation would larger than 1

Now take $g(x,y) = \alpha(x) \beta(y)$ with $\alpha \in L^1(\Omega)$ once again we find

$$\int_\Omega \alpha(x) \beta(u_{n_k}(x)) \, dx \to \int_\Omega \alpha(x) \left(\int_K \beta(y) \, d\mu_x(y) \right) dx$$

which proves (ii)

the essation (iii) follows in a similar way so we leave it as an exercise (!!)! To prove (iv) consider the function

$$g(x,y) = \alpha(x) y$$

and again

$$\int_\Omega \alpha(x) u_{n_k}(x) \to \int_\Omega \alpha(x) \left(\int_K y \, d\mu_x(x) \right) dx$$

holds for all $\alpha \in L^1(\Omega)$.

Finally to prove (v) we consider the function

$$f(x,y) = d_K(y, u_\infty(x)) \quad \text{where}$$

d_K is the distance on K. It follows that

$$\int_\Omega d_K(u_{n_k}(x), u_\infty(x)) \, dx \to \int_\Omega \left(\int_K d_K(y, u_\infty(x)) \, d\mu_x \right) dx$$

and since u_{n_k} converges in measure to u_∞ we can use lebesgue's dominated c thm and infer that

$$0 = \int_\Omega \left(\int_K d_K(y, u_\infty(x)) \, d\mu_x(y) \right) dx$$

in other word μ_x is supported on $\{u_\infty(x)\}$

the other implication is ok $\square$

Rmk If K is replaced by $\mathbb{R}^m$ we lose compactness $\to$ thus we consider the one point compactification of $\mathbb{R}^m$ and use it to produce a μ s.t.

$$\mu_x \in \mathcal{P}(\mathbb{R}^m \cup \{\infty\})$$

- The Young measure is the restriction of μ_x to $\mathbb{R}^m$ $\quad \bar{\mu}_x := \mu_x \llcorner_{\mathbb{R}^m}$

In general $\bar{\mu}_x$ is a sub-probability measure. However if one puts additional assumptions on the sequence u_n then it is possible to show that there is no mass at infinity and hence $\bar{\mu}_x$ is a probability measure.

RELAXATION WITH SEMICONTINUITY

Let $u_n : \Omega \to K$ be a sequence and let $x \mapsto \mu_x$ the Young measure associated to a specific subsequence of u_n

Lemma Let $\mathcal{L} : \Omega \times K \to [0, +\infty)$ be a Borel fct. s.t. $\mathcal{L}(x, \cdot)$ is l.s.c. for a.e. $x \in \Omega$ and set $\mathcal{E}(u) = \int \mathcal{L}(x, u)\, dx$

Then

$$\liminf \mathcal{E}(u_n) \geq \int_\Omega \left(\int_K \mathcal{L}(x, y)\, d\mu_x(y) \right) dx$$

proof (Sketch) Write $\mathcal{L}$ as a pointwise supremum of an increasing sequence $\phi_i : \Omega \times K \to [0, \infty)$ of Borel functions s.t. $\phi_i(x, \cdot)$ is cont. for a.e. $x \in \Omega$. We can thus apply (i) of previous thm. find that

$$\int \mathcal{L}(x, u_n)\, dx \geq \int \phi_i(x, u_n)\, dx \longrightarrow \int_\Omega \int_K \phi_i(x, y)\, d\mu_x(y) \Big) dx$$

Rmk If we assume moreover that $K \subset \mathbb{R}^m$ and $\mathcal{L}(x, \cdot)$ is cvx for all a.e. x then lemma(1) together with Jensen's inequality allows us to infer that

$$\liminf \mathcal{E}(u_n) \geq \int_\Omega \mathcal{L}\Big(x, \int_K y\, d\mu_x(y)\Big) = \int_\Omega \mathcal{L}(x, u_\infty(x))\, dx$$

thm Let $\mathcal{L} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to [0, +\infty)$ be a Borel fct s.t. $\mathcal{L}(x, \cdot, \cdot)$ is l.s.c. for a.e. $x \in \Omega$ and $\mathcal{L}(x, a, \cdot)$ convex for a.e. $x \in \Omega$ and all $a \in \mathbb{R}^m$. Then the functional $\mathcal{E}(u) = \int \mathcal{L}(x, u, Du)\, dx$ is weak-* l.s.c. on $W^{1,\infty}(\Omega; \mathbb{R}^m)$

proof Let $u_n \xrightarrow{*} u_\infty$ and consider the Young measure $x \mapsto \lambda_x \in \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^{m \times d})$ generated by the sequence

$$v_n = (u_n, Du_n)$$

By sobolev embedding $u_n \to u_\infty$ strongly (i.e. pointwise a.e.)

so λ_x is a Dirac mass on the first variable

$$\lambda_x = \delta_{u_\infty(x)} \otimes \mu_x$$

Then

$$\liminf \mathcal{E}(u_n) \geq \int_\Omega \left(\int_{\mathbb{R}^m \times \mathbb{R}^{m \times d}} \mathcal{L}(x, z, p) \, d\lambda(z, p) \right) dx$$

$$= \int_\Omega \left(\int_{\mathbb{R}^{m \times d}} \mathcal{L}(x, u_\infty(x), p) \, d\mu_x(p) \right) dx$$

$$\underset{\text{Jensen}}{\geq} \int_\Omega \mathcal{L}(x, u_\infty(x), \nabla u_\infty(x)) \, dx$$

the barycenter of μ_x is given by ∇u_∞