# CHAPTER IV : SOME REGULARITY RESULTS

## (IV.1) THE MODEL CASE : DIRICHLET ENERGY

Let $\Omega \subset \mathbb{R}^d$ be bounded open set with Lip. boundary and $g \in W^{1,2}(\Omega)$ then consider the following pb.

$$(P) \qquad \inf \left\{ \mathcal{E}(u) := \frac{1}{2}\int |\nabla u|^2 \;\middle|\; u - g \in W_0^{1,2}(\Omega) \right\}$$

One can show that $\exists!\ \bar{u}$ solution to (P). Furthermore $\bar{u}$ satisfies

$$(ELE) \int_\Omega \langle \nabla \bar{u}, \nabla \varphi \rangle \, dx = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega)$$

We will show that $\bar{u} \in C^\infty(\Omega)$ and satisfies the Laplace eq.

$$\Delta u = 0 \qquad \forall x \in \Omega$$

We speak then of interior regularity. If $\Omega$ is $C^\infty$ smooth and $g \in C^\infty(\bar{\Omega})$ one can show $\bar{u} \in C^\infty(\bar{\Omega})$ regularity up to the boundary.

Two proofs of regularity : one specific to the context the other one which could be applied to more general pb.

**thm** (Weyl Lemma) Let $\Omega \subset \mathbb{R}^d$ be open and $u \in L^1_{loc}(\Omega)$ sati.

$$\int u(x)\, \Delta v(x)\, dx = 0 \qquad \forall v \in C_c^\infty(\Omega) \qquad (*)$$

then $u \in C^\infty$ and $\Delta u = 0$ in $\Omega$

**Rmk** A solution of ELE satisfies $(*)$

**proof** Let $x \in \Omega$ and $R > 0$ suff. small so that

$$B_R(x) = \{ y \mid |y - x| < R \} \subset \Omega$$

Let $\sigma_{d-1} = \operatorname{meas}(\partial B_1(0))$

the idea is to show that

$$\bar{u}(x) = \frac{1}{\omega_{d-1} R^{d-1}} \int_{\partial B_R(x)} u \, d\sigma \qquad (\square)$$

then $\bar{u}$ is indipendent of $R$, $\bar{u} \in C(\Omega)$ and $\bar{u} = u$ a.e. in $\Omega$

<u>step 1</u> We start by making an appropriate choice of $v$ in $(*)$

let $R$ as above and choose $\varepsilon \in (0, R)$ and $\varphi \in C^\infty(\mathbb{R})$

with $supp(\varphi) \subset (\varepsilon, R)$ Define then

$$v(y) = \varphi(|x-y|) \quad \text{and observe}$$

that $v \in C_c^\infty(B_R(x))$

For $r = |x-y|$ we have

$$\Delta \varphi = \varphi''(r) + \frac{d-1}{r} \varphi'(r) = r^{1-d} \frac{d}{dr} [r^{d-1} \varphi'(r)]$$

From now on we only discuss the case $d \geq 2$

$$\text{let } \psi(r) = \frac{d}{dr} [r^{d-1} \varphi'(r)] \qquad (1)$$

and note that $\psi \in C_0^\infty(\varepsilon, R)$ and $\int_\varepsilon^R \psi(r) = 0$ $\quad (2)$

Remark that the converse is also true; namely that given $\psi \in C_0^\infty(\varepsilon, R)$ satisfying $(2)$ we can find $\varphi \in C_0^\infty(\varepsilon, R)$ verifying $(1)$.

<u>step 2</u> let $\psi \in C_0^\infty(\varepsilon, R)$ s.t. $(2)$ be arbitrary. Define the $\varphi$ and $v$ as above and use such $v$ in $(*)$ We get

Since $v \equiv 0$ on $\Omega \setminus B_R(x)$

$$0 = \int u \Delta v = \int_{B_R(x)} u \Delta v = \int_\varepsilon^R \psi(r) \underbrace{r^{1-d} \int_{\partial B_r(x)} u \, d\sigma}_{w(r)} \, dr$$

$$= \int_\varepsilon^R \psi(r) w(r) \, dr$$

and we get $w(r) = \text{const.}$ a.e. $r \in (\varepsilon, R)$

We denote this constant by $\omega_{d-1} \bar{u}(x)$ and by using the fact that $\varepsilon$ is arbitrary to write

$$w(r) = \gamma_{d-1}\, \bar{u}(x) \quad a.e. \quad r \in (0,R)$$

Step 3 First observe that the above equation is nothing else than ($\square$) with the fact that $\bar{u}$ is independent of $R$.
We integrate ($\square$) and get

$$\bar{u}(x) = \frac{1}{\text{meas}\,(B_R(x))} \int_{B_R(x)} u(s)\, ds \qquad (3)$$

From this we deduce that $\bar{u} \in C(\Omega)$. Let $x, y \in \Omega$ and $R$ be suff. small so that $\bar{B}_R(x) \cup \bar{B}_R(y) \subset \Omega$ we then have

$$|\bar{u}(x) - \bar{u}(y)| = \frac{1}{\omega_d R^{d}} \left| \int_{B_R(x)} u(s)\, ds - \int_{B_R(y)} u(s)\, ds \right|$$

$$\leq \frac{1}{\omega_d R^d} \int_O |u(s)| ds \quad \text{where} \quad O = (B_R(x) \cup B_R(y)) \setminus B_R(x) \cap B_R(y)$$

One can show that $\forall \varepsilon > 0 \ \exists \delta$ s.t. $\text{meas}(E) \leq \delta \Rightarrow \int_E |u(x)| dx \leq \varepsilon$
we deduce that $\bar{u}$ is cont.
It remains to prove that $\bar{u} = u$ a.e. $\Omega$. This follows from Leb. thm and the fact $u \in C^1_{loc}(\Omega)$. Indeed letting $R \to 0$ in (3) we have that for a.e. $x \in \Omega$ the right hand side of (3) is $u(x)$

thm let $k \geq 0$ an integer, $\Omega \subset \mathbb{R}^d$ be a bounded open set with $C^{k+2}$ boundary, $f \in W^{k,2}(\Omega)$ and

$$(P') \quad \inf \left\{ \mathcal{E}(u) := \int \frac{1}{2} |\nabla u|^2 + fu \;\middle|\; u \in W^{1,2}_0(\Omega) \right\}$$

then $\exists\, \bar{u} \in W^{k+2,2}(\Omega)$. $\exists\, \gamma = \gamma(\Omega, k) > 0$ so that

$$\|\bar{u}\|_{W^{k+2,2}} \leq \gamma \|f\|_{W^{k,2}}$$

In particular if $k = \infty$ then $\bar{u} \in C^\infty(\bar{\Omega})$

**Rmk** (i) Problem (P) and (P') are equivalent. If in (P) the boundary datum $g \in W^{k+2,2}(\Omega)$, then choose $f = \Delta g \in W^{k,2}(\Omega)$

(ii) A similar result can be obtained in Hölder spaces under appropriate reg. hypotheses on the boundary and $\beta \in (0,1)$

$$\|\bar{u}\|_{C^{k+2,\beta}} \leq \gamma \|f\|_{C^{k,\beta}}$$

If $\alpha \in (1, +\infty)$ it can also be proved that

$$\|\bar{u}\|_{W^{k+2,\alpha}} \leq \gamma \|f\|_{W^{k,\alpha}}$$

these are then known as Calderon-Zygmund estimates are considerably harder to obtain than those for $\alpha = 2$!

(iii) Both above results are however false if $\beta = 0$, $\beta = 1$ or $\alpha = \infty$ and if $\alpha = 1$. This is another reason why when dealing with partial ~~derivat~~ diff. eq. or the calculus of variations, Sobolev spaces and Hölder spaces are more appropriate than $C^k$ spaces.

<u>proof</u> We know from all the existence theory we have developed that (P') has a unique solution $\bar{u} \in W_0^{1,2}(\Omega)$ which satisfies in addition

$$\int_\Omega \langle \nabla \bar{u}, \nabla v \rangle \, dx = \int f v \, dx \qquad \forall v \in W_0^{1,2}(\Omega).$$

We will only show the <u>interior</u> <u>regularity</u> of $\bar{u}$, more precisely we will show that $f \in W^{k,2}(\Omega)$ implies $\bar{u} \in W_{loc}^{k+2,2}(\Omega)$ to show the sharper result we refer the interested student to thm 8.13 of Gilbarg-Trudinger.

The claim is then equivalent to proving that $\varphi\bar{u} \in W^{k+2,2}(\Omega)$ for every $\varphi \in C_c^\infty(\Omega)$. We let $u = \varphi\bar{u}$ and notice that $u \in W^{1,2}_{\mathbb{R}^d}(\Omega)$ and that it is a weak solution of

$$\Delta u = \Delta(\varphi\bar{u}) = \varphi\Delta\bar{u} + \bar{u}\Delta\varphi + 2\langle\nabla\bar{u},\nabla\varphi\rangle$$
$$= -\varphi f + \bar{u}\Delta\varphi + 2\langle\nabla\bar{u},\nabla\varphi\rangle = g$$

Since $f \in W^{k,2}(\Omega)$, $\bar{u} \in W_0^{1,2}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ we have then $g \in L^2_{\mathbb{R}^d}(\Omega)$

We have therefore transformed the problem into showing that any $u \in W^{1,2}(\mathbb{R}^d)$ which satisfies

$$\int \langle\nabla u, \nabla\varphi\rangle = \int g\,v \qquad \forall v \in W^{1,2}(\mathbb{R}^d)$$

is in fact $W^{2+k,2}(\mathbb{R}^d)$ whenever $W^{k,2}(\mathbb{R}^d)$

Step 1 We show that $g \in L^2(\mathbb{R}^d)$ implies $u \in W^{2,2}(\mathbb{R}^d)$. To achieve this goal we use the method of difference quotients. We introduce the following notations: for $h \in \mathbb{R}^d$, $h \neq 0$ we let

$$(D_h u)(x) = \frac{u(x+h) - u(x)}{|h|}$$

it easily follows from the characterization of Sobolev spaces that

$$\nabla(D_h u) = D_h(\nabla u)$$
$$\|D_{-h} u\|_{L^2} \leq \|\nabla u\|_{L^2}$$

$$\|D_h u\|_{L^2} \leq \gamma \implies u \in W^{1,2}$$

where $\gamma$ is a constant independent of $h$.

Choose now

$$v(x) = \big(D_{-h}(D_h u)\big)(x) = \frac{2u(x) - u(x+h) - u(x-h)}{|h|^2}$$

and observe that since $u \in W^{1,2}$ then $v \in W^{1,2}$

We therefore find

$$\int_{\mathbb{R}^d} \langle \nabla u, \nabla(D_{-h}(D_h(u)))\rangle = \int g(x)\, D_{-h}(D_h(u))\, dx$$

let us look at LHS

$$\int_{\mathbb{R}^d} \langle \nabla u, \nabla(D_{-h}(D_h(u)))\rangle\, dx = \frac{1}{|h|^2}\int \langle \nabla u, 2\nabla u - \nabla u(x-h) - \nabla u(x+h)\rangle\, dx$$

$$= \frac{2}{|h|^2}\int |\nabla u|^2 - \langle \nabla u, \nabla u(x+h)\rangle dx = \frac{1}{|h|^2}\int |\nabla u(x+h) - \nabla u|^2\, dx$$

where we have used

$$\int \langle \nabla u, \nabla u(x-h)\rangle = \int \langle \nabla u, \nabla u(x+h)\rangle$$

$$\int |\nabla u(x+h)|^2 = \int |\nabla u(x)|^2\, dx$$

So we have just found

$$\int \langle \nabla u, \nabla(D_{-h}(D_h(u)))\rangle = \frac{1}{|h|^2}\int |\nabla u(x+h) - \nabla u(x)|^2\, dx$$

$$= \int |D_h(\nabla u)|^2\, dx = \int g(x)\, D_{-h}(D_h(u))\, dx$$

Applying C-S. and the properties of $D_h$ we get

$$\|D_h \nabla u\|_{L^2}^2 \leq \|g\|_{L^2}\, \|D_{-h}(D_h(u))\|_{L^2}$$
$$\leq \|g\|_{L^2}\, \|D_h \nabla u\|_{L^2}$$

and hence

$$\|D_h \nabla u\|_{L^2} \leq \|g\|_{L^2}$$

We have indeed obtained $\nabla u \in W^{1,2}(\mathbb{R}^d)$ and hence $u \in W^{2,2}(\mathbb{R}^d)$ !

step 2 this step relies heavily on the form of the equation!
Let now $g \in W^{1,2}(\mathbb{R}^d)$ and let us show that $u \in W^{3,2}(\mathbb{R}^d)$.
The general case $g \in W^{k,2}$ implying that $u \in W^{2+k,2}$ follows by repeating the argument.
The idea is simple it consists in applying the previous step to $u_{x_i} = \frac{\partial u}{\partial x_i}$ and observing that since $\Delta u = g$ then $\Delta u_{x_i} = g_{x_i}$

For every $i = 1, \dots, d$

$$\int \langle \nabla u_{x_i}, \nabla v \rangle = \int g_{x_i} v \qquad \forall v \in W^{1,2}(\mathbb{R}^d)$$

it is sufficient to establish it for $v \in C_0^\infty(\mathbb{R}^d)$.

$$\int \langle \nabla u_{x_i}, \nabla v \rangle = \int \langle (\nabla u)_{x_i}, v \rangle = - \int \langle \nabla u, (\nabla v)_{x_i} \rangle \, dx$$

$$= - \int \langle \nabla u, \nabla v_{x_i} \rangle = - \int g \, v_{x_i} = \int g_{x_i} v$$

Since $g \in W^{1,2}$ then $g_{x_i} \in L^2$ and by the previous step we have $u_{x_i} \in W^{2,2}$. Since it holds for every $i = 1, \dots, d$ we get $u \in W^{3,2}$. ☒

## (IV.2) SOME GENERAL RESULTS

The generalization of previous regularity results to lagrangian of the form $\mathcal{L}(x, u, \nabla u)$ is a difficult task. However we give here the corresponding theorem without proof.

thm let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $\mathcal{L} \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^d)$
Let $\mathcal{L}$ satisfy for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$

$$\gamma_1 V^\alpha - \gamma_2 \leq \mathcal{L}(x, z, p) \leq \gamma_3 V^\alpha$$

$$|D_p \mathcal{L}|, |D_{xp} \mathcal{L}|, |D_z \mathcal{L}|, |D_{xz} \mathcal{L}| \leq \gamma_3 V^{\alpha - 1}$$

$$|D_{zp} \mathcal{L}|, |D_{zz} \mathcal{L}| \leq \gamma_3 V^{\alpha - 2}$$

$$\gamma_4 V^{\alpha-1}|\lambda|^2 \leq \sum_{i,j=1}^{d} D_{p_ip_j}\mathcal{L}(x,z,p) \leq \gamma_5 V^{\alpha-2}|\lambda|^2$$

where $\alpha \geq 2$, $V^2 = 1 + z^2 + |p|^2$ and $\gamma_i > 0$ $i = \cancel{0},\dots,5$ are constants.

Then any minimizer of

$$(P) \quad \inf\left\{ \mathcal{E}(u) := \int \mathcal{L}(x,u,\nabla u)\,dx \;\middle|\; u-g \in W_0^{1,\alpha}(\Omega)\right\}$$

is in $C^\infty(D)$ for every $D \subset \bar{D} \subset \Omega$

Rmk The last hyp. implies a kind of uniform convexity of $p \mapsto \mathcal{L}(x,z,p)$. It guarantees the uniform ellipticity of the E-L eq.

The proof of this theorem relies on the De Giorgi-Nash-Moser theory. During the proof one transforms the non linear ELE into an elliptic linear equation with bounded measurable coefficients.

Therefore to obtain the desired regularity one needs to know the regularity of solutions of such equations and this is precisely the famous them first established by De Giorgi then simplified by ~~Na~~ Moser and also proved at the same time but independently by Nash.

Thm Let $\Omega \subset \mathbb{R}^d$ open bounded set and $v \in W^{1,2}(\Omega)$ be a sol. of

$$\sum \int [a_{ij}(x)\, v_{x_i}(x)\, \varphi_{x_j}(x)]\,dx = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega)$$

where $a_{ij} \in L^\infty(\Omega)$ and denoting by $\gamma > 0$ a constant

$$\sum a_{ij}(x)\lambda_i\lambda_j \geq \gamma|\lambda|^2 \quad \text{a.e. in } \Omega \quad \forall \lambda \in \mathbb{R}^d$$

Then $\exists\, 0 < \beta < 1$ so that $v \in C^{0,\beta}(D)$ for every $D \subset \bar{D} \subset \Omega$.

Rmk It is interesting to understand at least formally the relationship between the two theorems in the case in which $\mathcal{L} = \mathcal{L}(p)$. The coefficients $a_{ij}(x)$ and the fct $v$ are respectively $D_{p_i p_j}\mathcal{L}(\nabla u)$ and $u_{x_i}$. The fact $v = u_{x_i} \in W^{1,2}$ is proved by the method of difference quotients