Université Paris-Saclay 2021-2022

Calculus of Variations
Final exam

Warnings :

- Duration : 3 hours.
- Electronic devices (including cell phones) are forbidden.
- The exercises are independent of each other.
- Answers must be fully justified and written in a rigorous manner.
- If a result of the course is used, it must be clearly mentioned.

Exercise 1. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous and convex function such that

$$h(t) \geq c|t|^q \quad \text{for all } t \in \mathbb{R},$$

for some $c > 0$ and $q > 1$, and let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We define the functional $F : W^{1,2}(\Omega; \mathbb{R}^2) \to [0, +\infty]$ by

$$F(u) = \int_\Omega \left[|\nabla u|^2 + h(\det \nabla u) \right] dx \quad \text{for all } u \in W^{1,2}(\Omega; \mathbb{R}^2).$$

1) Show that if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $W^{1,2}(\Omega; \mathbb{R}^2)$ such that

$$\sup_{n\in\mathbb{N}} F(u_n) < \infty,$$

then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,2}(\Omega; \mathbb{R}^2)$ and $\det\nabla u_n \rightharpoonup \det\nabla u$ weakly in $L^q(\Omega; \mathbb{R}^2)$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$.

2) Let $w \in W^{1,p}(\Omega; \mathbb{R}^2)$ be such that $F(w) < \infty$. We consider the minimization problem :

$$\inf_{v \in w + W^{1,2}_0(\Omega;\mathbb{R}^2)} F(v).$$

Using the direct method in the calculus of variations, show that this minimization problem admits solutions.

Exercise 2. Let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a continuous function such that

$$|f(\xi)| \leq C(1 + |\xi|^p) \quad \text{for all } \xi \in \mathbb{R}^{d\times N},$$

for some $C > 0$ and $1 \leq p < \infty$. We recall that the quasiconvexification of f is defined by

$$Qf(\xi) := \inf_{\varphi \in \mathcal{C}^\infty_c((0,1)^N;\mathbb{R}^d)} \int_{(0,1)^N} f(\xi + \nabla\varphi(y))\, dy.$$

Our goal is to show that

$$Qf(\xi) = \inf_{\varphi \in \mathcal{C}^\infty_{\text{per}}((0,1)^N;\mathbb{R}^d)} \int_{(0,1)^N} f(\xi + \nabla\varphi(y))\, dy,$$

where $\varphi \in \mathcal{C}^\infty_{\text{per}}((0,1)^N;\mathbb{R}^d)$ means that $\varphi : \mathbb{R}^N \to \mathbb{R}^d$ is of class $\mathcal{C}^\infty$ and φ is $(0,1)^N$-periodic.

Define

$$\bar{f}(\xi) := \inf_{\varphi \in \mathcal{C}^\infty_{\text{per}}((0,1)^N;\mathbb{R}^d)} \int_{(0,1)^N} f(\xi + \nabla\varphi(y))\, dy.$$

1. Show that $\bar{f} \le Qf$.
2. Let $\varphi \in \mathcal{C}^\infty_{\rm per}((0,1)^N;\mathbb{R}^d)$ and set $\varphi_j(x) := \frac{1}{j}\varphi(jx)$ for $j \in \mathbb{N}^*$ and $x \in \mathbb{R}^N$. Show that $\varphi_j \to 0$ strongly in $L^\infty((0,1)^N;\mathbb{R}^d)$ and that $|\nabla\varphi_j|^p \rightharpoonup \int_{(0,1)^N} |\nabla\varphi(y)|^p\,dy$ weakly* in $L^\infty((0,1)^N)$.
3. For all $t \in (0,1/2)$, we define $Q_t := (t, 1-t)^N$ and set $Q := Q_0$. For all $\delta < t < 1/2$, we consider a cut-off function $\zeta \in \mathcal{C}^\infty_c(Q)$ such that $0 \le \zeta \le 1$ and $|\nabla\zeta| \le C/\delta$ in Q, and $\zeta = 0$ on $Q \setminus Q_{t-\delta}$, $\zeta = 1$ on $Q_{t+\delta}$. Defining $\psi_j = \zeta\varphi_j$, show that for all $\xi \in \mathbb{R}^{d\times N}$,

$$Qf(\xi) \le \int_Q f(\xi + \nabla\psi_j)\,dx \le \int_Q f(\xi + \nabla\varphi_j)\,dx + f(\xi)|Q \setminus Q_{t-\delta}| + \frac{C}{\delta^p}\int_{L_\delta} |\varphi_j|^p\,dx + C\int_{L_\delta} (1 + |\xi|^p + |\nabla\varphi_j|^p)\,dx,$$

   where $L_\delta = Q_{t-\delta} \setminus Q_{t+\delta}$ and $C > 0$ is a constant independent of j, δ and t.
4. Deduce that

$$Qf(\xi) \le \int_Q f(\xi + \nabla\varphi(y))\,dy + f(\xi)|Q \setminus Q_{t-\delta}| + C\left(1 + |\xi|^p + \int_Q |\nabla\varphi(y)|^p\,dy\right)\mathcal{L}^N(L_\delta).$$

5. Show that

$$Qf(\xi) \le \int_Q f(\xi + \nabla\varphi(y))\,dy.$$

6. Deduce that $Qf \le \bar{f}$.

Exercise 3. Let $p > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz boundary and $(f_\varepsilon)_{\varepsilon>0}$ be a family of Carathéodory functions $f_\varepsilon : \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty)$ satisfying the coercivity and growth conditions

$$\lambda|\xi|^p \le f_\varepsilon(x,\xi) \le \Lambda(1 + |\xi|^p) \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^{d\times N} \text{ and all } \varepsilon > 0,$$

for some $0 < \lambda < \Lambda < +\infty$. Let $F_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ be the functional defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x,\nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d),\\ +\infty & \text{if } u \in L^p(\Omega;\mathbb{R}^d) \setminus W^{1,p}(\Omega;\mathbb{R}^d). \end{cases}$$

According to the abstract Γ-convergence result, there is no loss of generality to assume that there is a subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that F_{ε_j} Γ-converges in $L^p(\Omega;\mathbb{R}^d)$ to some functional $F : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ given by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d),\\ +\infty & \text{if } u \in L^p(\Omega;\mathbb{R}^d) \setminus W^{1,p}(\Omega;\mathbb{R}^d), \end{cases}$$

for some Carathéodory function $f : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$.

Let $\phi \in W^{1,p}(\Omega;\mathbb{R}^d)$ be fixed, define G^ϕ_ε and $G^\phi : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ by

$$G^\phi_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x,\nabla u)\,dx & \text{if } u \in \phi + W^{1,p}_0(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G^\phi(u) = \begin{cases} \displaystyle\int_\Omega f(x,\nabla u)\,dx & \text{if } u \in \phi + W^{1,p}_0(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}$$

Part I. The object of this part is to show that $G^\phi_{\varepsilon_j}$ Γ-converges in $L^p(\Omega;\mathbb{R}^d)$ to G^ϕ. In the sequel, we denote by G' and G'' the Γ-lower and the Γ-upper limits of $G^\phi_{\varepsilon_j}$, respectively.

1. Show that the lower bound $G^\phi \le G'$ holds.
2. We now intend to prove the upper bound $G^\phi \ge G''$ through the construction of a recovery sequence. Let $u \in \phi + W^{1,p}_0(\Omega;\mathbb{R}^d)$.
   (a) Why does there exists a sequence $(u_j)_{j\in\mathbb{N}}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$ such that $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$ and $\int_\Omega f_{\varepsilon_j}(x,\nabla u_j)\,dx \to \int_\Omega f(x,\nabla u)\,dx$?
   (b) Let $(\mu_j)_{j\in\mathbb{N}}$ be the sequence of nonnegative Radon measures in Ω defined by

   $$\mu_j = (1+|\nabla u|^p + |\nabla u_j|^p)\mathcal{L}^N.$$

   Show that there exists a subsequence (not relabeled) and a nonnegative Radon measure $\mu \in \mathcal{M}(\Omega)$ such that $\mu_j \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega)$.
   (c) For all $t>0$ small, let $\Omega_t = \{x\in\Omega : \ \mathrm{dist}(x,\partial\Omega) > t\}$. For all $0<\delta<t$, we consider a cut-off function $\zeta \in \mathcal{C}^\infty_c(\Omega)$ satisfying $0\le\zeta\le 1$ and $|\nabla\zeta| \le 2/\delta$ in Ω, $\zeta = 0$ in $\Omega\setminus\Omega_{t-\delta}$ and $\zeta = 1$ in $\Omega_{t+\delta}$. Define $v_j := \zeta u_j + (1-\zeta)u$. Show that v_j belongs to $\phi + W^{1,p}_0(\Omega;\mathbb{R}^d)$ and that $v_j \to u$ strongly in $L^p(\Omega;\mathbb{R}^d)$.
   (d) Show that

   $$\int_\Omega f_{\varepsilon_j}(x,\nabla u_j)\,dx \ge \int_\Omega f_{\varepsilon_j}(x,\nabla v_j)\,dx - \frac{C}{\delta^p}\int_{L_\delta}|u_j-u|^p\,dx - C\mu_j(L_\delta) - C\int_{\Omega\setminus\Omega_{t-\delta}}(1+|\nabla u|^p)\,dx,$$

   where $L_\delta = \Omega_{t-\delta}\setminus\Omega_{t+\delta}$ and $C>0$ is a constant independent of j, δ and t.
   (e) Deduce that

   $$\int_\Omega f(x,\nabla u)\,dx \ge \limsup_{j\to+\infty}\int_\Omega f_{\varepsilon_j}(x,\nabla v_j)\,dx - C\mu(\overline{L_\delta}) - C\int_{\Omega\setminus\Omega_{t-\delta}}(1+|\nabla u|^p)\,dx.$$

   (f) Prove first that for all $0<\delta<t$,

   $$G^\phi(u) \ge G''(u) - C\mu(\overline{L_\delta}) - \Lambda\int_{\Omega\setminus\Omega_{t-\delta}}(1+|\nabla u|^p)\,dx,$$

   then that for all $t>0$ small,

   $$G^\phi(u) \ge G''(u) - C\mu(\partial\Omega_t) - \Lambda\int_{\Omega\setminus\Omega_t}(1+|\nabla u|^p)\,dx,$$

   and finally that

   $$G^\phi(u) \ge G''(u).$$

   (g) Conclude.

3. Show that

$$\lim_{j\to+\infty} \inf_{v\in\phi+W_0^{1,p}(\Omega;\mathbb{R}^d)} \int_\Omega f_{\varepsilon_j}(x,\nabla v)\,dx = \min_{v\in\phi+W_0^{1,p}(\Omega;\mathbb{R}^d)} \int_\Omega f(x,\nabla v)\,dx.$$

Part II. In this second part, we show additional properties of the limit density f. For simplicity, we assume that $\Omega = (0,1)^N$ is the unit cube.

1. Show that if $(x,\xi) \mapsto f_\varepsilon(x,\xi)$ is independent of x, then so is f.
2. Prove that f is quasiconvex.
3. Show that $Qf_{\varepsilon_j}(\xi) \to f(\xi)$ for all $\xi \in \mathbb{R}^{d\times N}$.