Calculus of Variations – M2 AMS/Opt 2024/2025 – Université Paris-Saclay

Final Exam 17/01/2025

Duration : 2h15

- Do not hesitate to present partial solutions or ideas even when you are not sure how to formulate them rigorously.
- **Printed** documents are allowed.
- You can answer either in French or in English.
- If a result of the course is used, it must be clearly mentioned.
- **Exercises 1-3** deal with one dimensional CalVa; **Exercises 4-5** focus on higher dimensional problems.

Exercise 1 (3 points)**.** Consider the following functional

$$\mathcal{E}(u) := \int_0^1 ((1-|u'|^2)^2 + u^2)\mathrm{d}x$$

and the minimization problem

$$\min\{\mathcal{E}(u) \mid u \in W^{1,1}(0,1),\ u(0) = u(1) = 0\}. \tag{1}$$

By using the function $\varphi(x) = 1/2 - |x - 1/2|$ on $[0,1]$ and periodically extended to all $\mathbb{R}$, define a sequence u_h and prove that the above minimization problem does not have a solution.

Exercise 2 (5 points)**.** Consider the following functional

$$\mathcal{E}(u) := \int_0^1 (W(u') + u)\mathrm{d}x,$$

where $W(x) = (1-|x|^2)^2$ and the minimization problem

$$\min\{\mathcal{E}(u) \mid u \in W^{1,1}(0,1),\ u(0) = a,\ u(1) = b\}. \tag{2}$$

We want to show that it admits a solution for every $a, b \in \mathbb{R}$. In order to do it consider the following functional

$$\mathcal{G}(u) = \int_0^1 (V(u') + u)\mathrm{d}x,$$

where $V(x) = ((1-x^2)_-)^2$ (i.e $V = W$ outside $[-1,1]$ and $V = 0$ on $[-1,1]$) and the associated minimization problem

$$\min\{\mathcal{G}(u) \mid u \in W^{1,1}(0,1),\ u(0) = a,\ u(1) = b\}. \tag{3}$$

1. Prove that (3) admits a minimizer $\overline{u}$.
2. Find the Euler-Lagrange equation and deduce that $|\overline{u}'(x)| \geqslant 1$ almost everywhere.
3. Justify why $V = W^{**}$ and deduce that $\min \mathcal{G} \leqslant \min \mathcal{E}$.
4. Prove that $\overline{u}$ is indeed a minimizer to (2)

Exercise 3 (8 points)**.** Consider for every $\alpha \in \mathbb{R}$ and $p > 1$ the following functional

$$\mathcal{E}_{\alpha,p}(u) := \int_0^1 x^\alpha |u'|^p \mathrm{d}x$$

defined for every $u \in W^{1,1}(0,1)$ and the associated minimization problem

$$\min\{\mathcal{E}_{\alpha,p}(u) \mid u \in W^{1,1}(0,1),\ u(0) = a\ \ u(1) = b\}, \tag{4}$$

where a, b are two real numbers with $a \neq b$. We are going to study the existence for this problem for different cases.

1. Take $\alpha = p = 2$, then by using the sequence $u_h(x) = a + (b-a)\frac{\arctan(xh)}{\arctan(h)}$ show that the minimization problem above does not admit any solution whenever $a \neq b$.
2. Prove that when $\alpha \leqslant 0$ the problem admits a unique solution and find it.
3. Prove the existence and uniqueness of the solution when $\alpha > 0$ and $p > \alpha + 1$.
4. When $\alpha > 0$ and $p \leqslant \alpha + 1$, prove that no solution to (4) is possible when $a \neq b$. Consider the sequence $u_\varepsilon(x) = a + (b-a)\frac{\log(1+x/\varepsilon)}{\log(1+1/\varepsilon)}$.

Exercise 4 (8 points)**.** Consider a smooth bounded and connected domain $\Omega \subset \mathbb{R}^d$, a function $f \in L^2(\Omega)$ with $⨍ f = 0$, a number $\alpha \in \mathbb{R}$, and the minimization problem

$$\min\left\{\int_\Omega (1 + |\nabla u|)^2 + \cos(\alpha u + |\nabla u|) + fu \mid u \in H^1(\Omega)\right\}. \tag{5}$$

1. For $\alpha = 0$ prove that the minimization can be restricted to the function u with $⨍ u = 0$ and for $\alpha \neq 0$ to the functions u with $|⨍ u| \leqslant \frac{\pi}{|\alpha|}$.
2. Prove that the problem admits a solution.
3. Prove that any solution satisfies $\alpha \int \sin(\alpha u + |\nabla u|) = 0$.
4. Prove that the solution is unique up to additive constants if $\alpha = 0$.
5. Find all the minimizers in the case $f = 0$ and $\alpha \neq 0$.

Exercise 5 (6 points)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary, $p \in (1, +\infty)$ and $F_n : L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be a sequence of functionals such that for all $n \in \mathbb{N}$,

$$F_n(u) = +\infty \quad \text{if } u \in L^p(\Omega; \mathbb{R}^d) \setminus W^{1,p}(\Omega; \mathbb{R}^d),$$

and

$$\lambda \int_\Omega |\nabla u|^p \mathrm{d}x \leqslant F_n(u) \leqslant \Lambda \int_\Omega (1 + |\nabla u|^p) \mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega; \mathbb{R}^d),$$

where $0 < \lambda < \Lambda < +\infty$.

1. Assume that the sequence $(F_n)_{n\in\mathbb{N}}$ $\Gamma-$converges in $L^p(\Omega;\mathbb{R}^d)$ to some $F : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$.

(a) Show that $F(u) = +\infty$ for all ; $u \in L^p(\Omega;\mathbb{R}^d) \setminus W^{1,p}(\Omega;\mathbb{R}^d)$.

(b) Show that for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, then

$$\lambda \int_\Omega |\nabla u|^p \mathrm{d}x \leqslant F(u) \leqslant \Lambda \int_\Omega (1 + |\nabla u|^p)\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega;\mathbb{R}^d).$$

(c) Show that for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and all sequences $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$, then

$$F(u) \leqslant \lim_{n\to+\infty} \inf F_n(u_n).$$

(d) Show that for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ there exists sequence $\overline{u}_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$, then

$$F(u) = \lim_{n\to+\infty} F_n(\overline{u}_n).$$

2. Conversely, show that if $F : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ is a functional satisfying properties (c) and (d) of the previous question, then the sequence $(F_n)_{n\in\mathbb{N}}$ $\Gamma-$converges in $L^p(\Omega;\mathbb{R}^d)$ to $F : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$.