Variational Convergence and Homogenisation

Γ- convergence

(I) General theory

We want to define a notion of cv, named Γ-cv, assuring
(i) cv of the minimizers
(ii) cv of the energy

Given a metric space (X,d) and a sequence of functionals $\mathcal{E}_j : X \to \mathbb{R} \cup \{+\infty\}$ we would like to define a limit $\mathcal{E}$ satisfying:

- If $u_j \in X$ s.t. $\mathcal{E}_j(u_j) = \inf_X \mathcal{E}_j$ and $u_j \to u$ then $\mathcal{E}(u) = \inf_X \mathcal{E}$
- $\inf_X \mathcal{E}_j = \mathcal{E}_j(u_j) \to \mathcal{E}(u) = \inf_X \mathcal{E}$

<u>def</u> (Γ-cv) We say that $(\mathcal{E}_j)_j$ Γ-cv to $\mathcal{E}$ if $\forall u \in X$:

i) $\forall \{u_j\}_j \in X$ s.t. $u_j \to u$ we have (Lower Bound) LB

$$\mathcal{E}(u) \leq \liminf \mathcal{E}_j(u_j)$$

ii) $\exists (\bar{u}_j)_j \in X$ s.t. $\bar{u}_j \to u$ and (Upper Bound) UB

$$\mathcal{E}(u) = \lim \mathcal{E}_j(\bar{u}_j)$$

then the fundamental result of Γ-cv is the following one

thm Assume $\mathcal{E}_j$ cv to $\mathcal{E}$. If $\{u_j\}_j$ is a sequence in X s.t. $u_j \to u$ and $\lim_j \mathcal{E}_j(u_j) = \lim_j \inf_X \mathcal{E}_j$ then u is a minimizer of $\mathcal{E}$ and moreover

$$\mathcal{E}(u) = \min_X \mathcal{E} = \lim_{j \to +\infty} \inf_X \mathcal{E}_j$$

proof

Since $u_j \to u$ we know that

$$\mathcal{E}(u) \leq \liminf_j \inf_X \mathcal{E}_j(u_j)$$

Moreover $\forall v \in X$ $\exists \{\bar{v}_j\}_j$ s.t. $v_j \to v$ and

$$\mathcal{E}_j(\bar{v}_j) \to \mathcal{E}(v)$$

Thus,

$$\liminf_j \inf_X \mathcal{E}_j \leq \lim_j \mathcal{E}_j(\bar{v}_j) \leq \mathcal{E}(v)$$

which shows that $\mathcal{E}(u) \le \mathcal{E}(v)$ $\forall v \in X$ and

$\mathcal{E}(u) = \min_X \mathcal{E} \le \liminf \mathcal{E}_j \le \mathcal{E}(v)$ and taking $v = u$

we obtain the result

$\square$

From the definition of Γ-cv we have the following properties

Rmk 1 If $\mathcal{E}_j$ Γ-cv to Γ then Γ is l.s.c. in X.

let $u^n \to u$. Since we have the lim-sup bound $\forall k \in \mathbb{N}$ we can find a sequence $\bar{u}^k_j \to u^k$ s.t.

$\lim_j \mathcal{E}_j(u^k_j) = \mathcal{E}(u^k)$. By recurrence we can build a non-decreasing sequence σ_k of integers s.t. $\forall k \in \mathbb{N}$ and $\forall j \ge \sigma_k$

$$d(\bar{u}^k_j, u^k) \le \frac{1}{k} \quad |\mathcal{E}_j(u^k_j) - \mathcal{E}(u^k)| \le \frac{1}{k}$$

Define $v_j = \bar{u}^k_j$ if $\sigma_k \le j \le \sigma_{k+1}$ s.t.

$v_j \to u$

$$\mathcal{E}(u) \le \liminf \mathcal{E}_j(v_j) \le \liminf_k \mathcal{E}_{\sigma_k}(v_{\sigma_k})$$

$$\le \liminf \mathcal{E}(u^k)$$

Rmk 2 If $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ then all subsequence $\mathcal{E}_{j_k} \xrightarrow{\Gamma} \mathcal{E}$. To show the lim-inf we consider a sequence $u_k \to u$ and define

$$\bar{u}_j = \begin{cases} u_{j_k} & \text{if } j = j_k \\ u & \text{if } j \neq j_k \quad \forall k \end{cases}$$

s.t. $\bar{u}_j \to u$ and

$$\mathcal{E}(u) \leq \liminf_j \mathcal{E}_j(\bar{u}_j) \leq \liminf_k \mathcal{E}_{j_k}(\bar{u}_{j_k}) = \liminf_k \mathcal{E}_{j_k}(u_k)$$

For the lim sup $\quad \mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ then $\exists \{\bar{u}_j\} \in X$ s.t. $\bar{u}_j \to u$

and $\quad \mathcal{E}(u) = \lim \mathcal{E}_j(\bar{u}_j) = \lim_k \mathcal{E}_{j_k}(\bar{u}_{j_k})$

Rmk 3 If $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ and $G : X \to \mathbb{R}$ is cont. then
$\mathcal{E}_j + G \xrightarrow{\Gamma} \mathcal{E} + G$

def $\forall u \in X$ we define the Γ-limits

$$\mathcal{E}'(u) = \inf \{ \liminf \mathcal{E}_j(u_j) \mid u_j \to u \}$$

$$\mathcal{E}''(u) = \inf \{ \limsup \mathcal{E}_j(u_j) \mid u_j \to u \}$$

we have then the following result

prop $\mathcal{E}_j$ Γ cv if and only if $\mathcal{E}' = \mathcal{E}''$ and in this case the Γ-limit $\mathcal{E} = \mathcal{E}' = \mathcal{E}''$.

proof Assume $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ and prove that $\mathcal{E} = \mathcal{E}' = \mathcal{E}''$

Notice that $\mathcal{E}' \le \mathcal{E}''$ so it is enough to show that $\mathcal{E} \le \mathcal{E}'$ and $\mathcal{E}'' \le \mathcal{E}$

From the LB we have $\forall u \in X$ and $\forall u_j \to u$ that

$$\mathcal{E}(u) \le \liminf \mathcal{E}_j(u_j)$$

it is easy to see $\mathcal{E}(u) \le \mathcal{E}'$.

By the UB $\exists \bar{u}_j \to u$ s.t.

$$\mathcal{E}(u) = \lim_j \mathcal{E}_j(\bar{u}_j) = \limsup_j \mathcal{E}_j(\bar{u}_j) \ge \mathcal{E}''(u)$$

Assume now $\mathcal{E}' = \mathcal{E}'' = \mathcal{E}$. We want to show $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$

By definition $\forall u \in X$ and $\forall u_j \to u$

$$\mathcal{E}(u) = \mathcal{E}'(u) \le \liminf \mathcal{E}_j(u_j)$$

We have now to show the UB in order to conclude

in particular we have to show that $\exists \bar{u}_j \to u$ s.t.

$$\limsup \mathcal{E}_j(\bar{u}_j) \le \mathcal{E}''(u) = \mathcal{E}(u).$$

Consider the case $\mathcal{E}''(u) < +\infty$. $\forall k \in \mathbb{N}$ we can find a sequence $u_j^k \to u$ s.t.

$$\limsup_j \mathcal{E}_j(u_j^k) \le \mathcal{E}''(u) + \frac{1}{k}$$

By recurrence build a sequence σ_n non-decreasing s.t. $\forall k \in \mathbb{N}$ $\forall j \geq \sigma_n$

$$d(u_j^k, u) \leq \frac{1}{k} \qquad \mathcal{E}_j(u_j^k) \leq \mathcal{E}''(u) + \frac{2}{k}$$

take $\bar{u}_j = u_j^k$ if $\sigma_n \leq j \leq \sigma_{n+1}$ s.t. $\bar{u}_j \to u$
and $\limsup \mathcal{E}_j(\bar{u}_j) \leq \mathcal{E}''(u)$ $\square$

In separable metric spaces Γ cv enjoys a nice compactness property

Prop Let (X, d) a sep. metric space and $\mathcal{E}_j : X \to \mathbb{R} \cup \{+\infty\}$ a sequence of functionals. Then $\exists$ $\mathcal{E}_{j_k}$ and $\mathcal{E} : X \to \mathbb{R} \cup \{+\infty\}$ s.t. $\mathcal{E}_{j_k} \xrightarrow{\Gamma} \mathcal{E}$

Proof Since X is sep. $\exists$ a countable $\{u_i\}$
And by diagonal extraction we can extract a subsequence $\mathcal{E}_{j_k}$ s.t.

$$\lim_k \inf_{U_i} \mathcal{E}_{j_k} \quad \text{exists} \quad \forall i \in \mathbb{N}$$

$\forall u \in X$ define the Γ-liminf $\mathcal{E}'$ and Γ-limsup $\mathcal{E}''$ then we have only to show that $\mathcal{E}' = \mathcal{E}''$. We always have $\mathcal{E}' \leq \mathcal{E}''$

let $u \in X$ and $i \in \mathbb{N}$ s.t. $u \in U_i$ if $u_k \to u_i$ then we have $u_k \in V_i$ for k large enough and

$$\liminf_k \inf_{U_i} \mathcal{E}_{j_k} \le \liminf_k \mathcal{E}_{j_k}(u_k)$$

$$\sup_{\{i \in \mathbb{N} \mid u \in U_i\}} \limsup_k \inf_{U_i} \mathcal{E}_{j_k} \le \mathcal{E}'(u)$$

Since U_i is a basis we have

$$\sup_{\{i \in \mathbb{N} \mid u \in U_i\}} \limsup_k \inf_{U_i} \mathcal{E}_{j_k} \ge \sup_n \limsup_k \inf_{d(u,v) \le \frac{1}{n}} \mathcal{E}_{j_k}(v)$$

$\forall n \in \mathbb{N}$ we can find $\sigma_n \in \mathbb{N}$ s.t. $\forall k \ge \sigma_n$ we have

$$\inf_{d(u,v) \le \frac{1}{n}} \mathcal{E}_{j_k}(v) \le \mathcal{E}'(u) + \frac{1}{n}$$

$\exists v_k^n \in X$ s.t. $d(u, v_k^n) \le \frac{1}{n}$ with.

$$\mathcal{E}_{j_k}(v_k^n) \le \mathcal{E}'(u) + \frac{2}{n}$$

Again $\bar{v}_k = v_k^n$ if $\sigma_n \le k \le \sigma_{n+1}$ $\bar{v}_k \to u$ and

$$\mathcal{E}''(u) \le \limsup_k \mathcal{E}_{j_k}(\bar{v}_k) \le \mathcal{E}'(u)$$

Prop The sequence $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ if and only if for all subsequence there exists a subsequence which Γ-cv to $\mathcal{E}$

Proof We have already seen that if $\mathcal{E}_j \xrightarrow{\Gamma} \mathcal{E}$ then for all $\mathcal{E}_{j_k}$ we have $\mathcal{E}_{j_k} \xrightarrow{\Gamma} \mathcal{E}$

Now assume that $\mathcal{E}_j \not\xrightarrow{\Gamma} \mathcal{E}$. It means

- $\exists$ either $\bar{u}_j \to u$ s.t. $\liminf \mathcal{E}_j(\bar{u}_j) < \mathcal{E}(u)$
- or $\limsup \mathcal{E}_j(u_j) > \mathcal{E}(u)$ $\forall u_j \to u$.

1st case we can extract a subsequence s.t.

$$\lim_k \mathcal{E}_{j_k}(\bar{u}_{j_k}) = \liminf \mathcal{E}_j(\bar{u}_j) < \mathcal{E}(u)$$

By hyp from this subsequence $\mathcal{E}_{j_k}$ we can extract another subsequence $\mathcal{E}_{j_{k_\ell}} \xrightarrow{\Gamma} \mathcal{E}$ and in particular

$$\mathcal{E}(u) \le \liminf_\ell \mathcal{E}_{j_{k_\ell}}(\bar{u}_{j_{k_\ell}}) = \lim_\ell \mathcal{E}_{j_{k_\ell}}(\bar{u}_{j_{k_\ell}}) = \lim_k \mathcal{E}_{j_k}(\bar{u}_{j_k})$$

2nd case $\exists$ a neighb U of u s.t. $\mathcal{E}(u) \le \limsup_j \inf_U \mathcal{E}_j$

We can extract a subsequence

$$\lim_k \inf_U \mathcal{E}_{j_k} = \limsup \inf_U \mathcal{E}_j .$$

By hyp. we can extract another subsequence $\mathcal{E}_{j_{k_\ell}}$ Γ-cv $\mathcal{E}$

$\exists\ \bar{u}_\ell \to u$ ($u_\ell \in U$ for ℓ large enough) s.t.

$$\mathcal{E}(u) = \lim_{\ell} \mathcal{E}_{j_{k_\ell}}(\bar{u}_\ell) \geq \lim_{\ell} \inf_{U} \mathcal{E}_{\sigma_{k_\ell}} = \lim_{k} \inf_{U} \mathcal{E}_{\sigma_{k_\ell}}$$

<u>Example</u> $X = L^2(\mathbb{R})$ $\quad f_n \in L^2(\mathbb{R})$

$$\mathcal{E}_n(u) = \begin{cases} \int \frac{1}{p} |\nabla u|^p + f_n u & W_0^{1,p} \\ +\infty & \text{otherwise} \end{cases}$$

Assume $f_n \rightharpoonup f$ then we have $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$

- (liminf) $u_n \to u$ in L^2 and
  $\mathcal{E}_n(u_n) \leq C$

  the $W^{1,p}$ norm of u_n is bounded
  $\nabla u_n \rightharpoonup \nabla u$ in L^p

  and by semicontinuity we $\int \frac{1}{p} |\nabla u|^p \leq \liminf$
  $+ \int f_n u_n \to \int f u$ since we have
  weak cv of f_n and strong of u

  $\liminf \mathcal{E}(u_n) \geq \mathcal{E}(u)$

- (limsup) take $u_n = u$ with $\mathcal{E}(u) < +\infty$