Lecture 2
Calculus of Variations in 1D

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1 Regularity and the Lavrientev phenomenon

Let us consider again the problem

$$\inf\{\mathcal{E}(u) \mid u \in \mathcal{U}\} = m \tag{1.1}$$

where $\mathcal{U} := \{u \in W^{1,\alpha}(a,b) \,|\, u(a) = A,\, u(b) = B\}$ and $\mathcal{E}(u) = \int_a^b \mathcal{L}(x, u, u')$ with $\mathcal{L}$ of class C^2.

Before focusing on some regularity issue for the 1 dimensional case, let us consider the following existence theorem without proving it (we will see it for the general case later). Assume that the following hypothesis are satisfied

(H1) there exist $\alpha > q \geqslant 1$ and $c_1 > 0$, $c_2, c_3 \in \mathbb{R}$ such that for every $(x,u,p) \in [a,b] \times \mathbb{R} \times \mathbb{R}$

$$\mathcal{L}(x,u,p) \geqslant c_1|p|^\alpha + c_2|u|^q + c_3,$$

we will see that this ensures existence (notice that this condition says that the Lagrangian has a polynomial growth).

(H2) for every $\delta > 0$ there exists $c(\delta)$ such that for every $(x,u,p) \in [a,b]\times[-\delta,\delta]\times\mathbb{R}$

$$|\mathcal{L}(x,u,p)|, |\nabla_u\mathcal{L}(x,u,p)|, |\nabla_p\mathcal{L}(x,u,p)| \leqslant c(\delta)(1+|p|^\alpha),$$

this ensures that any minimizer of (1.1) satisfied the Euler-Lagrange equations we have studied above.

(H3) for all $(x, u, p) \in [a, b] \times [-\delta, \delta] \times \mathbb{R}$

$$D^2_{pp}\mathcal{L}(x, u, p) > 0.$$

Theorem 1.1 (Existence)**.** *Let $\mathcal{L} \in C^2$ and satisfy (H1) and (H3). Assume that there exists u_0 such that $\mathcal{E}(u_0) < \infty$ then there exists a unique minimizer to* (1.1)

Remark 1.2. (i) Notice that for the existence the hypothesis (H3) can be replace by asking only convexity in the variable p.

(ii) The theorem easily applies to the case of Dirichlet energy $\mathcal{L}(x, u, p) = \frac{1}{2}|p|^2$ with $\alpha = 2$. And also tot he natural generalization

$$\mathcal{L}(x, u, p) = \frac{1}{\alpha}|p|^\alpha + F(x, u)$$

where F is continuous and bounded from below (as we have seen at the beginning of this long lecture)

1.1 Regularity

Lemma 1.3. *Let $\mathcal{L} \in C^2$ and satisfy (H1),(H2) and (H3). Then any minimizer Then any minimizer $\overline{u} \in W^{1,\alpha}(a, b)$ of* (1.1) *is in fact in $W^{1,\infty}(a, b)$, and the Euler-Lagrange equation holds almost everywhere, i.e.,*

$$\frac{d}{dx}\left[\nabla_p\mathcal{L}(x, \overline{u}, \overline{u}')\right] = \nabla_u\mathcal{L}(x, \overline{u}, \overline{u}'), \quad \textit{a.e. } x \in (a, b).$$

Proof. First, we know that the following equation holds:

$$\int_a^b \left[\nabla_u\mathcal{L}(x, \overline{u}, \overline{u}')v + \nabla_p\mathcal{L}(x, \overline{u}, \overline{u}')v'\right] dx = 0, \quad \forall v \in C_0^\infty(a, b).$$

We then divide the proof into two steps.

Step 1. Define:

$$\varphi(x) := \nabla_p\mathcal{L}(x, \overline{u}(x), \overline{u}'(x)) \quad \text{and} \quad \psi(x) := \nabla\mathcal{L}(x, \overline{u}(x), \overline{u}'(x)). \tag{1.2}$$

We easily see that $\varphi \in W^{1,1}(a, b)$ and that $\varphi'(x) = \psi(x)$ for almost every $x \in (a, b)$, which means that

$$\frac{d}{dx}\left[\nabla_p\mathcal{L}(x, \overline{u}, \overline{u}')\right] = \nabla_u\mathcal{L}u(x, \overline{u}, \overline{u}'), \quad \text{a.e. } x \in (a, b). \tag{1.3}$$

Indeed, since $\overline{u} \in W^{1,\alpha}(a, b)$, and hence $\overline{u} \in L^\infty(a, b)$, we deduce from (H2) that $\psi \in L^1(a, b)$. We also have from (1.2) that

$$\int_a^b \psi(x)v(x)\, dx = -\int_a^b \varphi(x)v'(x)\, dx, \quad \forall v \in C_0^\infty(a, b).$$

Since $\varphi \in L^1(a,b)$ (from (H2)), we have by the definition of weak derivatives the claim, namely $\varphi \in W^{1,1}(a,b)$ and $\varphi' = \psi$ a.e.

Step 2. Since $\varphi \in W^{1,1}(a,b)$, we have that $\varphi \in C^0([a,b])$, which means that there exists a constant $c_5 > 0$ such that:

$$|\varphi(x)| = |\nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x))| \leqslant c_5, \quad \forall x \in [a,b]. \tag{1.4}$$

Since $\overline{u}$ is bounded (and even continuous), let us say $|\overline{u}(x)| \leqslant \delta$ for every $x \in [a,b]$, we have from (H3) (notice that this hypothesis implies convexity of the lagrangian in p) that:

$$\mathcal{L}(x,u,0) \geqslant \mathcal{L}(x,u,p) - p\nabla_p \mathcal{L}(x,u,p), \quad \forall (x,u,p) \in [a,b] \times [-\delta,\delta] \times \mathbb{R}.$$

Combining this inequality with (H1), we find that there exists $c_6 \in \mathbb{R}$ such that, for every $(x,u,p) \in [a,b] \times [-\delta,\delta] \times \mathbb{R}$,

$$p\nabla_p \mathcal{L}(x,u,p) \geqslant \mathcal{L}(x,u,p) - \mathcal{L}(x,u,0) \geqslant c_1|p|^\alpha + c_6.$$

Using (1.4) and the above inequality, we find:

$$c_1|\overline{u}'|^\alpha + c_6 \leqslant \overline{u}'\nabla_p \mathcal{L}(x,\overline{u},\overline{u}') \leqslant |\overline{u}'||\nabla_p \mathcal{L}(x,\overline{u},\overline{u}')| \leqslant c_5|\overline{u}'|, \quad \text{a.e. } x \in (a,b),$$

which implies, since $\alpha > 1$, that $|\overline{u}'|$ is uniformly bounded. Thus, the lemma. □

Theorem 1.4. *Let $\mathcal{L} \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$ satisfy (H1), (H2), and (H3). Then any minimizer of* (1.1) *is in $C^\infty([a,b])$.*

Proof. We divide the proof into two steps.

Step 1. We know from Lemma 1.3 that

$$x \mapsto \varphi(x) := \nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x))$$

is in $W^{1,1}(a,b)$ and hence it is continuous. Consider now the Legendre transform of the function $\mathcal{L}$, that is

$$\mathcal{L}^*(x,u,v) := \sup_{\xi \in \mathbb{R}} \{vp - \mathcal{L}(x,u,\xi)\},$$

then $\mathcal{L}^* \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$ (we will show it in 2 lectures!) and, for every $x \in [a,b]$, we have

$$\varphi(x) = \nabla_p \mathcal{L}(x, \overline{u}(x), \overline{u}'(x)) \iff \overline{u}'(x) = \mathcal{L}^*_v(x, u(x), \varphi(x)).$$

Since $\nabla_v \mathcal{L}^*$, $\overline{u}$, and φ are continuous, we infer that $\overline{u}'$ is continuous and hence $\overline{u} \in C^1([a,b])$. We therefore deduce that $x \mapsto \nabla_u \mathcal{L}(x, u(x), u'(x))$ is continuous, which, combined with the fact that

$$\frac{d}{dx}[\varphi(x)] = \nabla_u \mathcal{L}(x, u(x), u'(x)), \quad \text{a.e. } x \in (a,b),$$

(or equivalently ,by properties of $\mathcal{L}^*$, $\varphi' = -\nabla_u \mathcal{L}^*(x,u,\varphi)$) leads to $\varphi \in C^1([a,b])$.

Step 2. Considering now the system:

$$\begin{cases} \overline{u}'(x) = \nabla_v \mathcal{L}^*(x, \overline{u}(x), \varphi(x)), \\ \varphi'(x) = -\nabla_u \mathcal{L}^*(x, \overline{u}(x), \varphi(x)), \end{cases}$$

we can start our iteration. Indeed, since $\mathcal{L}^*$ is C^∞ and $\overline{u}$ and φ are C^1, we deduce from our system that, in fact, $\overline{u}$ and φ are C^2. Returning to the system, we get that $\overline{u}$ and φ are C^3. Finally, we conclude that $\overline{u}$ is C^∞, as desired. □

1.2 The Lavrientev phenomenon

We have seen here that, under some assumptions on the growth of the Lagrangian function, we are able to prove existence results as well as the well-posedness of the Euler-Lagrangian equations for a "weak" minimizer (remember that we are working on Sobolev spaces). So it is quite natural to have the impression that we have found the right space to work with and the correct "generalization" of minimum problems involving an integral energy whose Lagrangian has a polynomial (superlinear indeed) growth. Unfortunately this is just an impression (as it is often the case in math!). If we want to consider the minimizer of problem (1.1) as a "generalized solution" of the problem *Minimize* $\mathcal{E}(u)$ *in the class of smooth functions with* $u(a) = A$ *and* $u(b) = B$ we should at least expect that the infimum of (1.1) agrees with the infimum on the class of smooth functions, i.e.

$$\inf_{u \in W^{1,1},\, u(a)=A,\, u(b)=B} \mathcal{E}(u) = \inf_{u\,\mathrm{smooth},\, u(a)=A,\, u(b)=B} \mathcal{E}(u).$$

Theorem 1.5 (Mania's example)**.** *Let*

$$\mathcal{L}(x,u,p) := (x-u^3)^2 p^6, \quad \mathcal{E}(u) := \int_0^1 \mathcal{L}(x,u(x),u'(x))\,dx.$$

Let

$$\mathcal{W}_\infty := \{u \in W^{1,\infty}(0,1) : u(0)=0,\ u(1)=1\},$$
$$\mathcal{W}_1 := \{u \in W^{1,1}(0,1) : u(0)=0,\ u(1)=1\}.$$

Then

$$\inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_\infty\} > \inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_1\} = 0.$$

Moreover, $u(x) = x^{1/3}$ *is a minimizer of* $\mathcal{E}$ *over* $\mathcal{W}_1$*.*

Lemma 1.6. *Let* $0 < a < b < 1$ *and*

$$\mathcal{W}_{a,b} := \{u \in W^{1,\infty}(a,b) \,|\, u(a) = \frac{1}{4}a^{1/3},\ u(b) = \frac{1}{2}b^{1/3},\ \frac{1}{4}x^{1/3} \leqslant u(x) \leqslant \frac{1}{2}x^{1/3}\ \forall x \in [a,b]\}.$$

If $\mathcal{L}(x,u,p) = (x-u^3)^2 p^6$ *and*

$$\mathcal{E}_{a,b}(u) := \int_a^b \mathcal{L}(x,u(x),u'(x))\,dx,$$

then

$$\mathcal{E}_{a,b}(u) \geqslant c_0 b,$$

for every $u \in \mathcal{W}_{a,b}$ *and for* $c_0 = 7^2 3^5 2{-}18 5^{-5}$.

Proof. Theorem 1.5 **Step 1** We first prove that if $u \in \mathcal{W}_\infty$, then there exist $0 < a < b < 1$ such that $u \in \mathcal{W}_{a,b}$, namely

$$\begin{cases} u(a) = \frac{1}{4}a^{1/3}, \\ u(b) = \frac{1}{2}b^{1/3}, \\ \frac{1}{4}x^{1/3} \leqslant u(x) \leqslant \frac{1}{2}x^{1/3},\ \forall x \in [a,b]. \end{cases} \tag{1.5}$$

The existence of such a and b is easily seen . Let

$$A := \{a \in (0,1) \,|\, u(a) = \frac{1}{4}a^{1/3}\},$$

$$B := \{b \in (0,1) \,|\, u(b) = \frac{1}{2}b^{1/3}\}.$$

Since u is Lipschitz, $u(0) = 0$, and $u(1) = 1$, it follows that $A \neq \emptyset$ and $B \neq \emptyset$. Next, choose

$$a := \max\{\alpha : \alpha \in A\}, \quad \beta := \min\{\beta : \beta \in B, \beta > a\}.$$

It is then clear that a and v satisfy the required (1.5).

Step 2 We may therefore use the lemma to deduce that, for every $u \in \mathcal{W}_\infty$,

$$\mathcal{E}(u) = \int_0^1 (x - u^3)^2 u'^6 \, dx \geqslant \int_a^b (x - u^3)^2 u'^6 \, dx \geqslant c_0 b > c_0 > 0.$$

Thus,

$$\inf\{\mathcal{E}(u) : u \in \mathcal{W}_\infty\} \geqslant c_0 > 0.$$

Step 3 The fact that $u(x) = x^{1/3}$ is a minimizer of $\mathcal{E}$ over all $u \in \mathcal{W}_1$ is trivial. Hence,

$$\inf\{\mathcal{E}(u) \,|\, u \in \mathcal{W}_1\} = 0.$$

This achieves the proof of the theorem.

□

2 Some reminders

2.1 Sobolev Spaces in 1D

Given an open interval $I \subset \mathbb{R}$ and an exponent $p \in [1, +\infty]$, we define the Sobolev space $W^{1,p}(I)$ as:

$$W^{1,p}(I) := \left\{ u \in L^p(I) : \exists g \in L^p(I) \text{ s.t. } \int u\varphi' \, dx = -\int g\varphi \, dx \text{ for all } \varphi \in C_c^\infty(I) \right\}.$$

The function g, if it exists, is unique, and will be denoted by u' since it plays the role of the derivative of u in the integration by parts.

The space $W^{1,p}$ is endowed with the norm:

$$\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|u'\|_{L^p}.$$

With this norm, $W^{1,p}$ is a Banach space, separable if $p < +\infty$, and reflexive if $p \in (1, \infty)$.

All functions $u \in W^{1,p}(I)$ admit a continuous representative, which moreover satisfies:

$$u(t_0) - u(t_1) = \int_{t_1}^{t_0} u'(x) \, dx.$$

This representative is differentiable a.e., and the pointwise derivative coincides with the function u' a.e. Moreover, for $p > 1$, the same representative is also Hölder continuous, of exponent $\alpha = 1 - \frac{1}{p} > 0$. The injection from $W^{1,p}$ into $C^0(I)$ is compact if I is bounded.

If $p = 2$, the space $W^{1,p}$ can be given a Hilbert structure, choosing as a norm:

$$\sqrt{\|u\|_{L^2}^2 + \|u'\|_{L^2}^2},$$

and is denoted by H^1.

Higher-order Sobolev spaces $W^{k,p}$ can also be defined for $k \in \mathbb{N}$ by induction as follows:

$$W^{k+1,p}(I) := \{u \in W^{k,p}(I) : u' \in W^{k,p}(I)\},$$

and the norm in $W^{k+1,p}$ is defined as $\|u\|_{W^{k,p}} + \|u'\|_{W^{k,p}}$. In the case $p = 2$, the Hilbert spaces $W^{k,2}$ are also denoted by H^k.

2.2 Hilbert Spaces

A Hilbert space is a Banach space whose norm is induced by a scalar product: $\|x\| = \sqrt{x \cdot x}$.

Theorem (Riesz): If H is a Hilbert space, for every $\xi \in H'$ there is a unique vector $h \in H$ such that $\langle \xi, x \rangle = h \cdot x$ for every $x \in H$, and the dual space H' is isomorphic to H.

In a Hilbert space H, we say that x_n weakly converges to x and we write $x_n \rightharpoonup x$ if $h \cdot x_n \to h \cdot x$ for every $h \in H$. Every weakly convergent sequence is bounded, and if $x_n \rightharpoonup x$, using $h = x$, we find:

$$\|x\|^2 = x \cdot x = \lim x \cdot x_n \leqslant \liminf \|x\| \|x_n\|,$$

i.e., $\|x\| \leqslant \liminf \|x_n\|$.

In a Hilbert space H, every bounded sequence x_n admits a weakly convergent subsequence.

2.3 Weierstrass Criterion for the Existence of Minimizers, Semicontinuity

The most common way to prove that a function admits a minimizer is called the *direct method in the calculus of variations*. It simply consists of the classical Weierstrass Theorem, possibly replacing continuity with semicontinuity.

Definition: On a metric space X, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous* (l.s.c. in short) if for every sequence $x_n \to x$, we have:

$$f(x) \leqslant \liminf f(x_n).$$

A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *upper semicontinuous* (u.s.c. in short) if for every sequence $x_n \to x$, we have:

$$f(x) \geqslant \limsup f(x_n).$$

Definition: A metric space X is said to be *compact* if from any sequence x_n, we can extract a convergent subsequence $x_{n_k} \to x \in X$.
Theorem (Weierstrass): If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and X is compact, then there exists an $\overline{x} \in X$ such that:

$$f(\overline{x}) = \min\{f(x) : x \in X\}.$$

Proof: Define $\# := \inf\{f(x) : x \in X\} \in \mathbb{R}\cup\{-\infty\}$ ($\# = +\infty$ only if f is identically $+\infty$, but in this case, any point in X minimizes f). By definition, there exists a minimizing sequence x_n, i.e., points in X such that $f(x_n) \to \#$. By compactness, we can assume $x_n \to \overline{x}$. By lower semicontinuity, we have:

$$f(\overline{x}) \leqslant \liminf f(x_n) = \#.$$

On the other hand, we have $f(\overline{x}) \geqslant \#$ since $\#$ is the infimum. This proves $\# = f(\overline{x}) \in \mathbb{R}$ and this value is the minimum of f, realized at $\overline{x}$.

2.4 Sobolev Spaces in Higher Dimensions

Given an open domain $\Omega \subset \mathbb{R}^d$ and an exponent $p \in [1, +\infty]$, we define the Sobolev space $W^{1,p}(\Omega)$ as:

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \forall i, \exists g_i \in L^p(\Omega) \text{ s.t. } \int_\Omega u \partial_{x_i}\varphi \, dx = -\int_\Omega g_i \varphi \, dx, \, \forall \varphi \in C_c^\infty(\Omega)\}.$$

Properties:

- The functions g_i, if they exist, are unique and form a vector denoted by ∇u, which plays the role of the derivative.
- The norm on $W^{1,p}(\Omega)$ is defined as:

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

- $W^{1,p}(\Omega)$ is a Banach space, separable if $p < \infty$, and reflexive if $p \in (1, \infty)$.

Functions in $W^{1,p}(\Omega)$ can also be characterized as those functions $u \in L^p(\Omega)$ whose translations satisfy:

$$\|u_h - u\|_{L^p(\Omega')} \leqslant C|h|,$$

for every subdomain $\Omega' \subset\subset \Omega$ and $|h|$ sufficiently small. The optimal constant C in this inequality equals $\|\nabla u\|_{L^p}$.

Sobolev Embeddings

- If $p < d$, then $W^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$, where $p^* = \frac{pd}{d-p}$.
- If $p = d$, then $W^{1,p}(\Omega)$ embeds compactly into $L^q(\Omega)$ for any $q < \infty$.
- If $p > d$, then all functions in $W^{1,p}(\Omega)$ admit a continuous representative, which is Hölder continuous with exponent $\alpha = 1 - \frac{d}{p}$.

2.5 Traces of Sobolev Functions

If Ω is a smooth domain and $p > 1$, the trace operator:

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega),$$

has the following properties:

- It is linear, continuous, and compact.
- For Lipschitz functions u, $\mathrm{Tr}[u] = u|_{\partial\Omega}$.
- The kernel of Tr is precisely $W_0^{1,p}(\Omega)$.
- For $p > d$, Tr maps into $C^{0,\alpha}(\partial\Omega)$, with $\alpha = 1 - \frac{d}{p}$.