Exam, Optimal Transport, 27 May 2014, 3 hours

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23 mars 2022

Exercise 1 (4 points)

Let μ be the Lebesgue measure on $[0,1]$ and ν be the Borel measure defined on $[0,1]$ by

$$\int_{[0,1]} f(y)\nu(dy) = (1-\alpha)\int_0^1 f(y)dy + \alpha f(1), \quad \forall f \in C(\mathbf{R}),$$

where α is fixed in $[0,1]$. What is the optimal transport map T sending μ to ν ? Find the values of α for which $T : [0,1] \to [0,1]$ is a smooth map (resp. a one-to-one map).

Exercise 2 (4 points)

Let μ be the Lebesgue measure on $[0,1]^2$ and let

$$T : (x,y) \in [0,1]^2 \to (\sqrt{x}\cos(2n\pi y), \sqrt{x}\sin(2n\pi y)) \in \mathbf{R}^2,$$

where n is a fixed integer $n \in \{0,1,2\}$. What is the image measure ν of μ by T ? Find the values of n for which
i) T is an optimal transport map ;
ii) ν is absolutely continuous with respect to the Lebesgue measure.

Problem (12 points)

Let D be the unit cube in $\mathbf{R}^d$. We are given two smooth functions $\gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$\int_D \gamma_0(x)dx = \int_D \gamma_1(x)dx = 1.$$

We denote $Q = [0,1] \times D$ and set, for every continuous real-valued function $((t,x) \to f(t,x)) \in C(Q)$,

$$BT[f] = \int_Q (f(1,x)\gamma_1(x) - f(0,x)\gamma_0(x))dx.$$

We are given a convex function $K : \ C(Q) \to \mathbf{R}$. We denote by $K^* : \ C(Q)' \to]-\infty, +\infty]$ the Legendre-Fenchel transform of K.
We consider the set $\mathbf{F}$ of all pairs of smooth real-valued functions (ϕ, θ) defined on $Q = [0,1] \times D$

$$(t,x) \in Q \to \phi(t,x) \in \mathbf{R}, \ \ (t,x) \in Q \to \theta(t,x) \in \mathbf{R},$$

such that

$$\partial_t \phi(t,x) + \frac{1}{2}|\nabla_x \phi(t,x)|^2 \leq \theta(t,x), \quad \forall (t,x) \in Q.$$

We consider the maximization problem

$$J = \sup_{(\phi,\theta)\in \mathbf{F}} BT[\phi] - K[\theta]$$

and a maximizing sequence $(\phi^\epsilon, \theta^\epsilon)$ such that $BT[\phi^\epsilon] - K[\theta^\epsilon] \geq J - \epsilon$, for $\epsilon \in]0,1]$.

Question 1 (6 points)

Using the Rockafellar duality theorem, show the existence of at least one pair of (Borel) measures (c, m) defined on Q and respectively valued in $\mathbf{R}_+$ and $\mathbf{R^d}$ such that : m is absolutely continuous with respect to c with density $v \in L^2(Q, dc\, ;\, \mathbf{R}^d)$, with

$$m(dt,dx) = v(t,x)c(dt,dx), \quad J = \int_Q \frac{1}{2}|v(t,x)|^2 dc(t,x) + K^*[c],$$

$$\int_Q (\partial_t f(t,x) + v(t,x)\cdot \nabla_x f(t,x))dc(t,x) = BT(f), \quad \text{for every smooth function f on Q},$$

$$\int_Q |\partial_t \phi^\epsilon(t,x) + \frac{1}{2}|\nabla_x \phi^\epsilon(t,x)|^2 - \theta^\epsilon(t,x)|dc(t,x) \leq \epsilon, \quad \int_Q \frac{1}{2}|v(t,x) - \nabla_x \phi^\epsilon(t,x)|^2 dc(t,x) \leq \epsilon,$$

$$0 \leq K^*[c] + K[\theta^\epsilon] - < c, \theta^\epsilon > \leq \epsilon.$$

Question 2 (3 points)

Assume that there exist optimal solutions (ϕ, θ, c, v) and that all of them satisfy $c(dt,dx) = \gamma(t,x)dtdx$, with ϕ, θ, γ, v smooth and $\gamma > 0$. With the help of the previous question, prove that these solutions satisfy the system of equations in the interior of $Q = [0,1] \times D$,

$$\partial_t \gamma(t,x) + \nabla_x \cdot (\gamma(t,x)v(t,x)) = 0,$$

$$\partial_t v(t,x) + (v(t,x)\cdot \nabla_x)v(t,x) = E(t,x)$$

(where the fields v et E will be respectively defined in terms of ϕ and θ), as well as the time-boundary conditions : $\gamma(0,\cdot) = \gamma_0, \;\; \gamma(1,\cdot) = \gamma_1$ on D. What can one say about the uniqueness of E ?

Question 3 (3 points)

Still in the framework of the previous question, we restrict ourself to the case :

$$K : \theta \in C(Q) \to \int_Q \exp(\theta(t,x))dtdx.$$

Show that

$$\theta(t,x) = \lambda(\gamma(t,x))$$

for a certain function $\lambda :]-\infty, +\infty[$ to be found. Then, discuss the uniqueness of (γ, v).